Sharing a collective probability of success

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Scenario

A group of actors have a target to reach.

Actors could be factors.

Each actor-factor has a probability of reaching the target.

They act *jointly* but *independently*.

Question: how to divide the resulting *collective* probability of success?

Applications: military, medical, legal,...

Hou–Xu–Sun–Driessen (*Operations Research Letters* 46, 2018) have considered this problem within the theory of *transferable utility games* and proposed to solve the question using the *Shapley value*.

Probability game

Data: a set $N = \{1,2,...,n\}$ of "players" each characterized by a probability of success, p_i for player i, $0 \leq p_i \leq 1$.

A pair (N,p) is a *probabilistic situation*.

A *transferable utility game* (N,u) is associated to a probabilistic situation.

The characteristic function u associates to every subset $S \subset N$ the probability $u(S)$ that coalition S succeeds.

Assuming *independence*, the probability that the players in $S \subset N$ *all* fail is given by:

$$\prod_{j \in S} (1 - p_j)$$

Hence...

$$u(S) = 1 - \prod_{j \in S} (1 - p_j) = \sum_{S \in \mathcal{C}(N)} (-1)^{t-1} \prod_{j \in S} p_j$$

is the probability that at least one player in S succeeds, *assuming that the players outside S do not participate.*

$$\mathcal{C}(N) = \{S \subset N \mid S \neq \emptyset\}$$

Dual probability game

The *dual* of the probability game (N,u) is the game (N,u^d) defined by:

$$u^d(S) = \prod_{j \in N \setminus S}(1-p_j) - \prod_{j \in N}(1-p_j) = \left(1 - \prod_{j \in S}(1-p_j)\right)\prod_{j \in N \setminus S}(1-p_j)$$

It is the probability that *at least one* member of S succeeds *assuming that all players outside S fail.*

$$v^d(S) = v(N) - v(N \setminus S)$$

In the 3-player case:

$$u(i) = p_i \quad i \in \{1, 2, 3\},$$

$$u(i, j) = 1 - (1 - p_i)(1 - p_j) = p_i + p_j - p_i p_j \quad i, j \in \{1, 2, 3\}, i \neq j,$$

$$u(1, 2, 3) = 1 - (1 - p_1)(1 - p_2)(1 - p_3) = p_1 + p_2 + p_3 - p_1 p_2 - p_1 p_3 - p_2 p_3 + p_1 p_2 p_3.$$

$$u^d(i) = p_i - p_i p_j - p_i p_k + p_1 p_2 p_3 \quad \text{for } i, j, k \in \{1, 2, 3\}, j, k \neq i, j \neq k$$

$$u^d(i, j) = p_i + p_j - p_1 p_2 - p_1 p_3 - p_2 p_3 + p_1 p_2 p_3 \quad \text{for } i, j \in \{1, 2, 3\}, i \neq j,$$

$$u^d(1, 2, 3) = u(1, 2, 3) = p_1 + p_2 + p_3 - p_1 p_2 - p_1 p_3 - p_2 p_3 + p_1 p_2 p_3.$$

Properties

Probability games and their duals are *monotonic*: $S \subset T \Rightarrow v(S) \leq v(T)$.

Probability games are *concave* (and thereby subadditive)

marginal contributions are decreasing: $i \in S \subset T \Rightarrow v(S) - v(S \setminus i) \geq v(T) - v(T \setminus i)$.

Dual probability games are *convex* (and thereby superadditive)

marginal contributions are increasing: $i \in S \subset T \Rightarrow v(S) - v(S \setminus i) \leq v(T) - v(T \setminus i)$.

The *collective* probability of success is given by:

$$u(N) = 1 - \prod_{j \in N} (1 - p_j)$$

$$= \sum_{T \in \mathcal{C}(N)} (-1)^{t-1} \prod_{i \in T} p_i = \sum_{i \in N} p_i - \sum_{\substack{i,j \in N \\ i \neq j}} p_i p_j + \sum_{\substack{i,j,k \in N \\ i \neq j,k \;\; j \neq k}} p_i p_j p_k - \ldots$$

$$= u^d(N)$$

We are interested in *allocating* to each player a share in this overall probability.

The core

By concavity, the core of a probability game is nonempty:

$$C(N,u)=\left\{x\in\mathbb{R}^n \,\middle|\, \sum_{i\in N} x_i = u(N) \text{ and } \sum_{i\in S} x_i \le u(S) \text{ for all } S\in\mathcal{C}(N)\right\}$$

By convexity, the core of the dual of a probability game is nonempty:

$$C(N,u^d)=\left\{x\in\mathbb{R}^n \,\middle|\, \sum_{i\in N} x_i = u^d(N) \text{ and } \sum_{i\in S} x_i \ge u^d(S) \text{ for all } S\in\mathcal{C}(N)\right\}$$

$$\mathcal{C}(N)=\{S\subset N \mid S\neq\varnothing\}$$

The core is *selfdual*: the core of a game coincides with the core of its dual. Indeed, core allocations x of a game (N,v) satisfy the following inequalities:

$$v(S) \leq \sum_{i \in S} x_i \leq v(N) - v(N \setminus S)$$

In the 3-player case, the core $C(N,p)$ associated to the probabilistic situation (N,p) is the set of allocations x such that:

$$p_1 - p_1 p_2 - p_1 p_3 + p_1 p_2 p_3 \leq x_1 \leq p_1$$

$$p_2 - p_1 p_2 - p_2 p_3 + p_1 p_2 p_3 \leq x_2 \leq p_2$$

$$p_3 - p_1 p_3 - p_2 p_3 + p_1 p_2 p_3 \leq x_3 \leq p_3$$

The lower bound on p_1 is the probability that 1 succeeds while {2,3} fails.

Harsanyi dividends

The set of games on N can be identified to $\mathbb{R}^{2^n-1}$. The *unanimity games* (N,u_T) defined by

$$u_T(S) = 0 \quad \text{if } T \subset S$$
$$= 1 \quad \text{if } T \not\subset S$$

form a *basis* of the set of games on the player set N.

For all game (N,v), there exists a unique 2^n-1 dimensional vector $\alpha(N,v) = (\alpha_T)_{T \in \mathcal{C}(N)}$ such that:

$$v(S) = \sum_{T \in \mathcal{C}(N)} \alpha_T u_T(S) = \sum_{T \in \mathcal{C}(S)} \alpha_T$$

α_T is interpreted as the (Harsanyi) *dividend* accruing to coalition T.

Dividends can be defined recursively, starting with $\alpha_\emptyset = 0$.

For a probability game (N,u): $\alpha_T(N,u) = (-1)^{t-1} \prod_{i \in T} p_i$

For a dual probability game (N,u^d): $\alpha_T(N,u^d) = \prod_{j \in T} p_j \prod_{j \in N \setminus T} (1 - p_j)$

It is the probability that all players in T succeed while players outside T all fail.

The $\alpha_T(N,u)$ alternate in sign according to coalition size while the $\alpha_T(N,u^d)$ are all *non-negative*.

This is the characteristic of *positive games*.

Positive games are convex and have interesting implications: solution concepts tend to converge on this class of games.

Allocation rules

For a given class of games Γ, an *allocation rule* is a mapping φ that associates an allocation to any given game in Γ.

Equal division is the simplest allocation rule: $ED_i(N,v) = \frac{1}{n} v(N)$

Equal division of the surplus is slightly more sophisticated rule:

$$EDS_i(N,v) = v(i) + \frac{1}{n}\left(v(N) - \sum_{j \in N} v(j) \right) \quad i = 1, \ldots, n.$$

These rules do not consider the differences in players' contributions.

The Shapley value

The sum of all dividends is equal to $v(N) \Rightarrow$ an allocation can be obtained by distributing the dividends of every coalition among its members.

The Shapley value divides *uniformly* the dividends among its members:

$$SV_i(N,v) = \sum_{T \in \mathcal{C}_i(N)} \frac{1}{t} \alpha_T(N,v)$$

$$\mathcal{C}_i(S) = \{S \subset N \,|\, i \in S\}$$

Applied to a probability game and its dual:

$$SV_i(N,u) = \sum_{T \in \mathcal{C}_i(N)} \frac{(-1)^{t-1}}{t} \prod_{j \in T} p_j = p_i \sum_{T \in \mathcal{C}_i(N)} \frac{(-1)^{t-1}}{t} \prod_{j \in T \setminus i} p_j$$

$$SV_i(N,u^d) = \sum_{T \in \mathcal{C}_i(N)} \frac{1}{t} \prod_{j \in T} p_j \prod_{j \in N \setminus T} (1 - p_j) = p_i \sum_{T \in \mathcal{C}_i(N)} \frac{1}{t} \prod_{j \in T / i} p_j \prod_{j \in N \setminus T} (1 - p_j)$$

The Shapley value is *selfdual*: $SV_i(N,u) = SV_i(N,u^d) = SV_i(N,p)$

The Shapley value allocates to a player a share *proportional* to his probability:

$$SV_i(N, p) = p_i f(p_{-i})$$

where the coefficient of proportionality is defined by the *player-independent* and *symmetric* function $f : \mathbb{R}_+^{n-1} \to \mathbb{R}_+$ defined by:

$$f(z) = 1 + \sum_{T \in \mathcal{C}(\{1,\ldots,n-1\})} \frac{(-1)^t}{t+1} \prod_{j \in T} z_j.$$

This property actually characterizes the Shapley value.

p_{-i} denotes the vector of probabilities of which the coordinate i has been eliminated.

An allocation rule φ on the set of probabilistic situations (N,p) satisfies *neutral proportionality* if there exists a symmetric function $f : \mathbb{R}_+^{n-1} \to \mathbb{R}_+$ such that

$$\varphi_i(N, p) = p_i f(p_{-i}) \text{ for all } i = 1, \ldots, n$$

There is *one and only one* allocation rule that satisfies neutral proportionality.

Then is must be the Shapley value.

The *weighted Shapley value* associated to given *positive* weights $w = (w_1,\ldots,w_n)$ is defined by:

$$SV_i(N,v,w) = \sum_{T\in\mathcal{C}_i(N)} \frac{w_i}{w(T)} \alpha_T(N,v)$$

where $w(T) = \sum_{i\in T} w_i$.

Applied to a probability situation (N,p):

$$SV_i(N,p,w) = p_i \sum_{T\in\mathcal{C}_i(N)} \frac{w_i}{w(T)} \prod_{j\in T/i} p_j \prod_{j\in N\setminus T} (1-p_j)$$

Proportionality still applies but (of course) not neutrality.

In general, the following sequence of inclusions hold:

$$C(N,v) \subset WS(N,v) \subset H(N,v)$$

where *WS* denotes the set of all weighted Shapley values and *H* denotes the set of all distributions of Harsanyi dividends.

For a positive game, all three coincide. As a consequence, we have:

$$C(N,p) = WS(N,p) = H(N,u^d) \subset H(N,u)$$

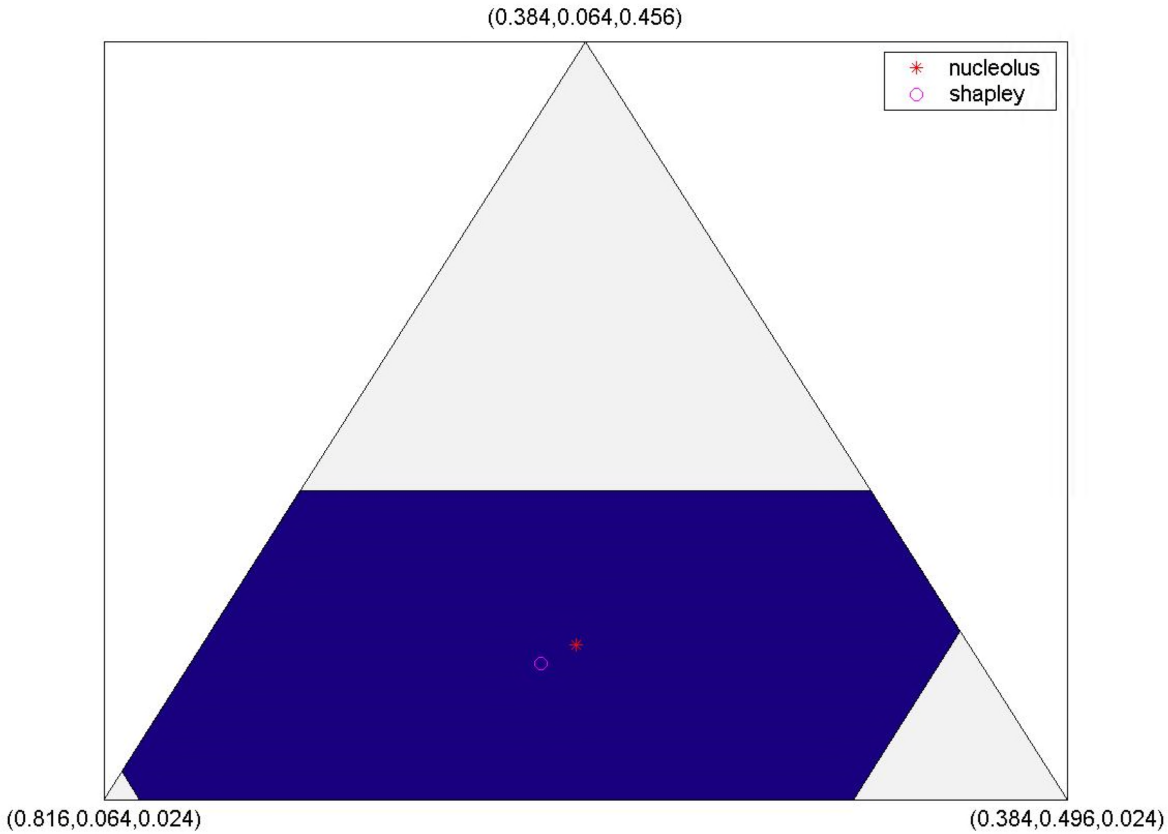
Example p = (0.8, 0.4, 0.2)

u = (0.8, 0.4, 0.2, 0.88, 0.84, 0.52, 0.904)

Probability to be divided = 0.904

Shapley shares:
(0.581, 0.221, 0.101)

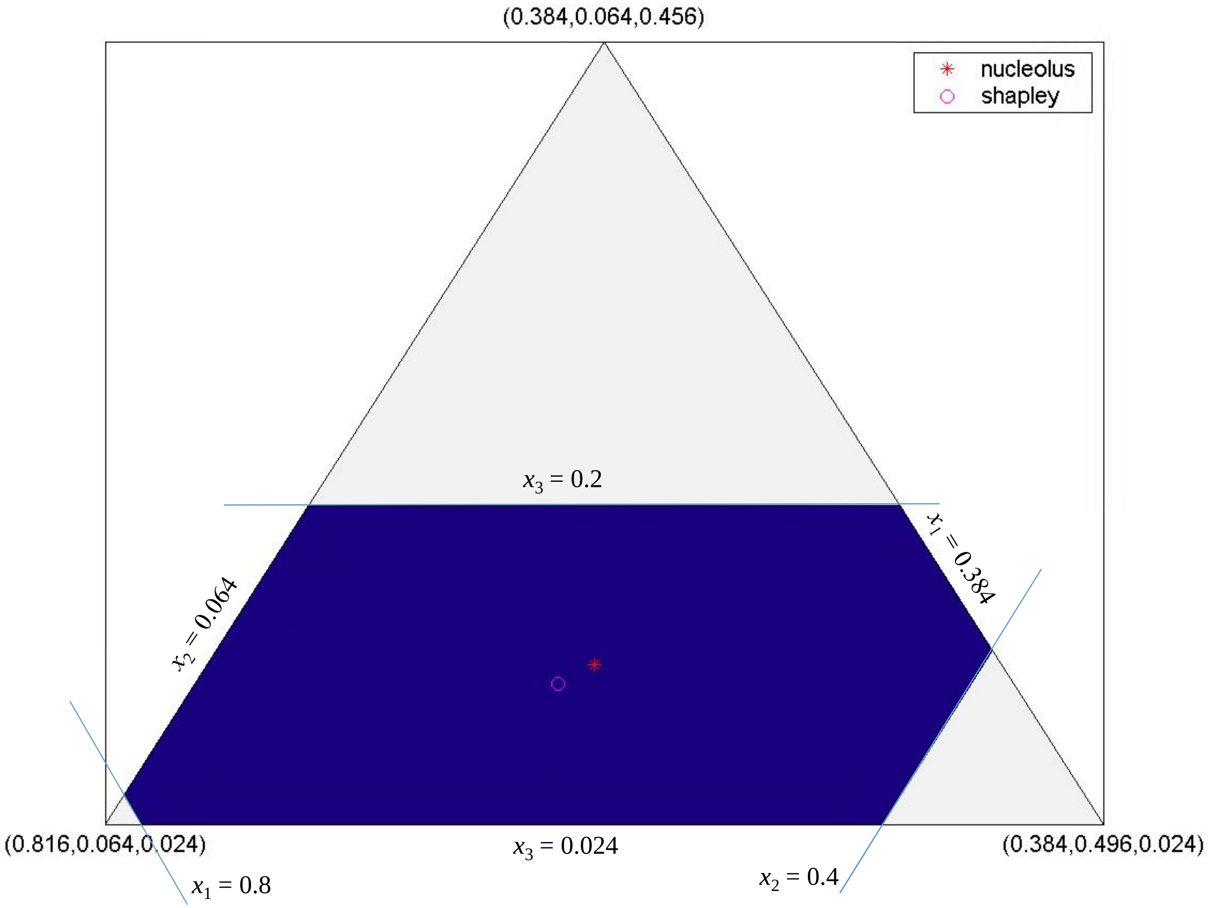
Normalized shares:
(0.643, 0.245, 0.112)



www.tuglabweb.uvigo.es (M.A. Mirás Calvo and E. Sánchez Rodríguez)

$u = (0.8, 0.4, 0.2, 0.88, 0.84, 0.52, 0.904)$

$$0.384 \leq x_1 \leq 0.8$$
$$0.064 \leq x_2 \leq 0.4$$
$$0.024 \leq x_3 \leq 0.2$$

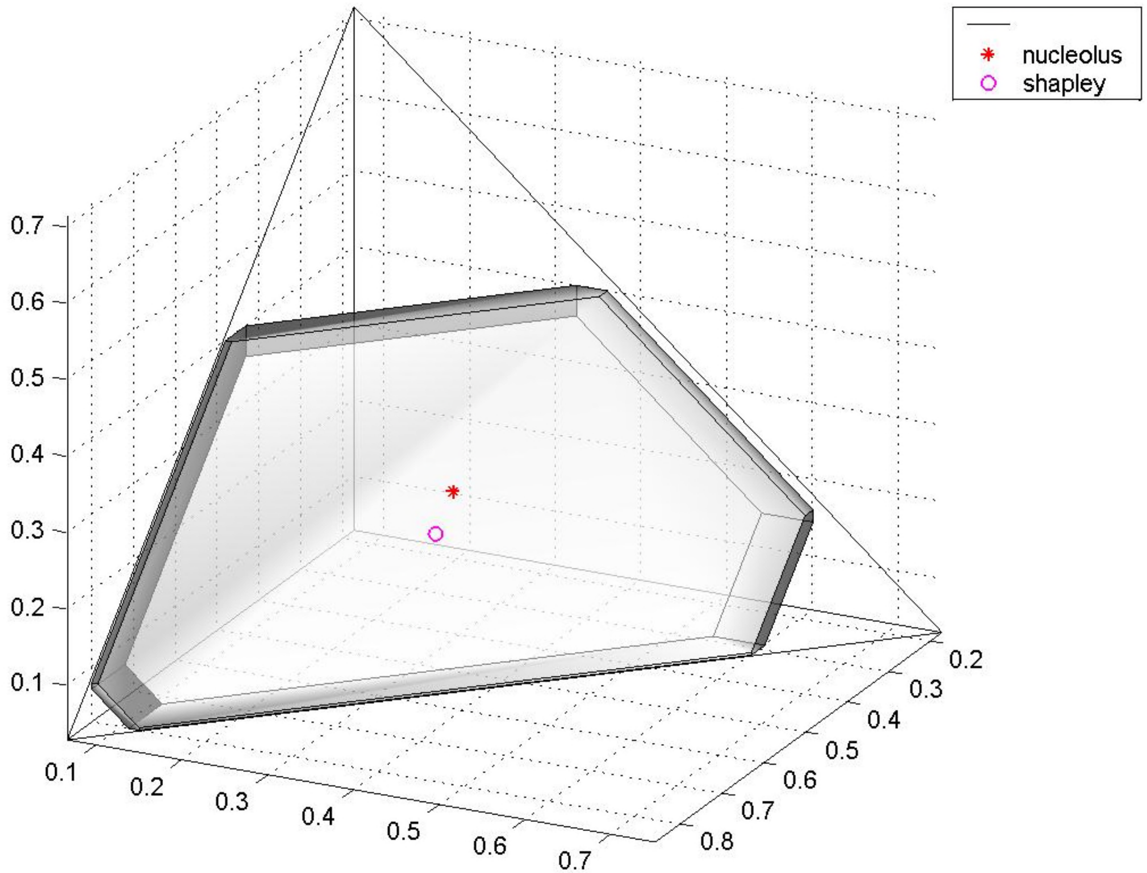


Example p = (0.8, 0.6, 0.4, 0.1)

Probability to be divided = 0.957

Shapley shares:
(0.446, 0.293, 0.178, 0.040)

Normalized shares:
(0.466, 0.306, 0.186, 0.042)



Concluding remarks

The case where $p_i = 1$ for some i.

The case where some p_i are small.

The nucleolus: a formula?

The assumption of independence, a limitation.

The end...

Thanks for your attention

Your comments and suggestions are most welcome.