Apportionment methods and practices

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Kempten Autumn Talks

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Part I

Apportionment problem

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The problem

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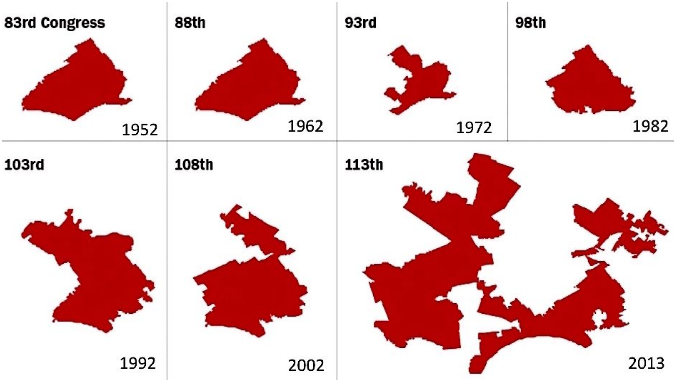
Two subproblems

- How to allocate seats among the regions? $\longleftarrow$ We look at this.
- How to draw the boundaries of the individual regions? $\longleftarrow$ Not this, although the manipulative design of voting districts, known as **gerrymandering** is an interesting topic in itself.

Gerry-salamander

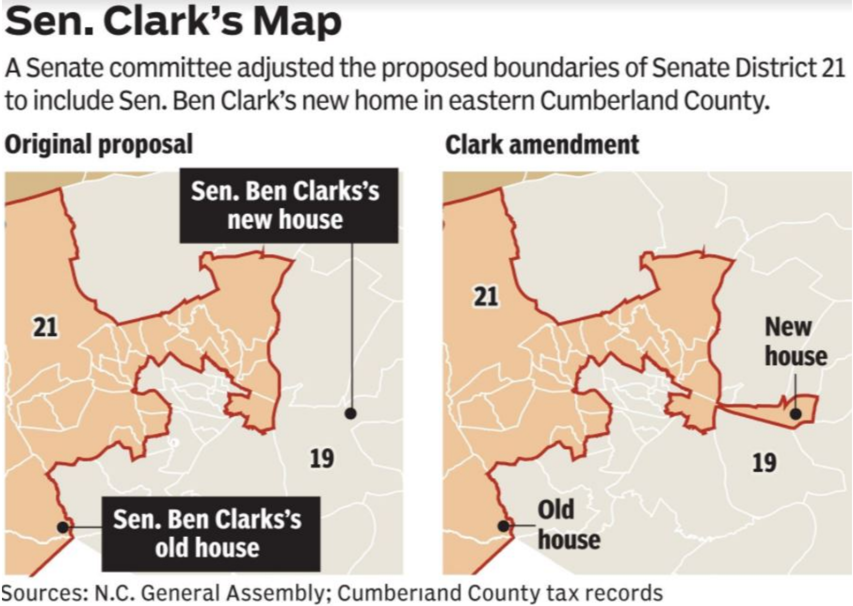
The electoral district boundaries under Governor Elbridge Gerry in 1812

Gerrymandering today



The shape of the 7th District in Pennsylvania develops strangely

Gerrymandering today



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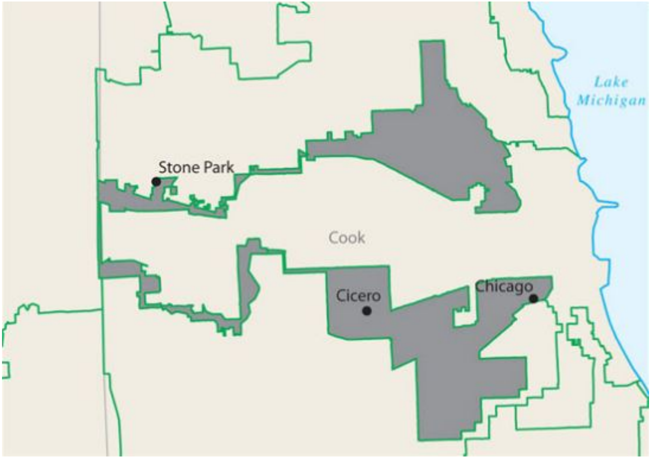
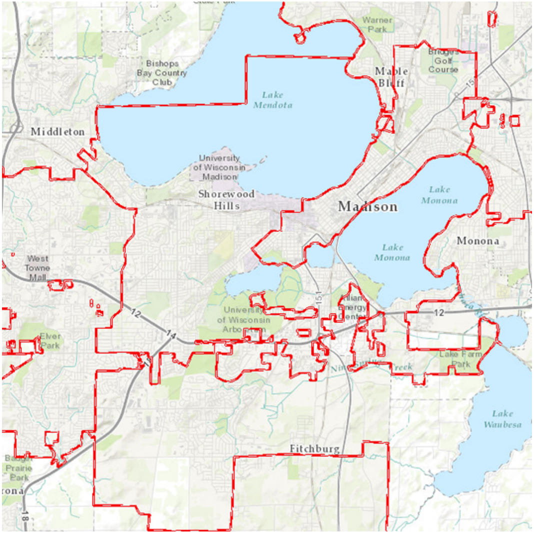


Illinois 4th congressional district was to have a majority-Hispanic district in the Chicago area, a Puerto Rican strip on the North and a Mexican-American on the South connected by a highway.

Gerrymandering today



A map of Wisconsin district boundaries.

(Back to apportionment)

Aside: Why representation?

- A recognition of separatist groupings
  - heterogeneous voters with locally homogeneous preferences.

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  - Have a single (multi-candidate) constituency (e.g. the Netherlands)
  - Partisan politics reduces problem.

Apportionment examples: trivial case

30000 persons 30000 persons

allocate 2 reps

Apportionment examples: trivial case


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1 rep	1 rep
30000 persons/rep	30000 persons/rep

30000 persons/rep on average

0% difference from average

Apportionment examples: harder case

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30000 persons　　30000 persons

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2 reps　　1 rep

15000 persons/rep　　30000 persons/rep

20000 persons/rep on average

-25%　　+50%

Apportionment examples: dilemma

20000 persons

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allocate 4 reps

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10000 persons/p 15000 persons/p

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1 rep 3 rep

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12500 persons/p on average

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Which allocation is preferred?

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Which allocation is preferred in general?

Historical introduction

The US was the first modern country that adopted sophisticated apportionment techniques ever since 1789.

> *"The Number of Representatives shall not exceed one for every thirty Thousand, but each State shall have at least one Representative; ..."* US Constitution, Art I, § 2, Cl 3

Not *apportionment* in the strict sense.

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1792	Jefferson method	Apportionment Act of 1792
1842	Webster method	Act of 25 June 1842, ch 46, 5 Stat
1850	Hamilton method	Act of 23 May 1850, 9 Stat. 432-433
1911	Webster method	Apportionment Act of 1911
1940	Huntington–Hill method	

Proportional apportionment

14th Amendment of the US Constitution (1868)

"Representatives shall be apportioned among the several states according to their respective numbers, counting the whole number of persons in each state, excluding Indians not taxed."

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Not universal!

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- Spain: 2 deputies per province, then proportional
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- Hong Kong: new territories (with low populations) are preferred
- There are *biased* and *distorted* methods.

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We focus on *proportional* apportionment.

The basis for apportionment

Basis for apportionment

"One man – one vote"

- ∴ based on actual voter turnout – how to predict?
- ∴ based on voter numbers – (used to be) hard to calculate
- ∴ based on population numbers

Originally §2 said

> *"... adding to the whole Number of free Persons ..., and excluding Indians not taxed, three fifths of all other Persons."*

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> *"... adding to the whole Number of free Persons ..., and excluding Indians not taxed, three fifths of all other Persons."*

∴ Indians neither taxed nor represented; slaves counted as $3/5$.

> *"But when the right to vote...is denied to any of the male inhabitants of such State,...the basis for representation therein shall be reduced..."*

∴ Voter restrictions reduce apportionment claim.

Apportionment problem

Apportionment

A fair division problem to distribute indivisible and indistinguishable objects among agents with heterogeneous claims.

Used for allocating

- districts among states (etc.) based on populations,
- seats among parties based on votes received in an election,
- both,
- schedule tasks (time slots are allocated), or
- resources to tasks in resource management (Kubiak, 2009).

Apportionment in fair representation

Difficulties

- Seats are indivisible: fractional seats cannot be allocated.
- The sizes of the constituencies should be roughly the same. Under ideal circumstances, every constituency contains the same number of voters.
- Constituency boundaries may be affected by the geography of the region, by administrative or historic boundary lines, or because of the concentration of a specific national minority.

Mathematical model

- Let $N = \{1, 2, \ldots, n\}$ denote the number of states.
- State population vector is given by $\boldsymbol{p} = (p_1, p_2, \ldots, p_n)$.
- Let $H \in \mathbb{N}$ be the size of the Parliament (aka House size).
- We seek positive integers $a_1, a_2, \ldots, a_n$ s.t. $a_1 + a_2 + \cdots + a_n = H$. The a_i denotes the number of seats that state i obtains.
- Let $P = p_1 + p_2 + \cdots + p_n$ be the total population, and let $A = \frac{P}{H}$ denote the average size of a constituency.

Definition

An apportionment method M is a mapping that assigns an allotment, $\mathbf{a} = (a_1, a_2, \ldots, a_n)$ for each apportionment problem $(\boldsymbol{p}, H)$.

Properties of apportionment methods

Properties and methods: Exact quota

Exact quota

If all the $\frac{p_i}{A}$ numbers are integers[a], then a sensible apportionment method M should assign $\frac{p_i}{A}$ seats to state i for every $i \in N$.

If a methods shows this feature we say it has the exact quota property.

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[a] Highly unlikely in practice, but can be tested.

Example

Let $p_1 = 100$ and $p_2 = 200$, furthermore let $H = 3$.
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What if quotas are not exact?

Properties and methods: Hare quota

In reality, population sizes seldom allow exact quota distributions. A logical generalization of exact quota is to require a method to assign rounded values:

Hare quota

For a general apportionment problem we may require that

- no state should get less than the lower integer part of the quota, $\lfloor \frac{p_i}{A} \rfloor$, or
- no state should get more than the upper integer part of the quota, $\leq \lceil \frac{p_i}{A} \rceil$, or
- both, that is, $\lfloor \frac{p_i}{A} \rfloor \leq M(\mathbf{p}, H) \leq \lceil \frac{p_i}{A} \rceil$ for all $i \in N$.

If a methods that abides these rules, we say it satisfies, respectively, the **Lower Hare Quota**, **Upper Hare Quota**, or **Hare Quota Property**.

Example

Let $p_1 = 120$ and $p_2 = 280$, and let $H = 4$. Then the average size of a constituency, $A = 100$. The quota of the fist state is $\frac{p_i}{A} = 2.8$, thus a method that satisfies Hare quota gives this State either 2 or 3 seats.

Largest remainder methods

Largest remainder methods – Hamilton method

Hamilton method

1. Distribute $\lfloor \frac{p_i}{A} \rfloor$ seats for each State i
2. Arrange the states in a decreasing order according to the values $d_i = \frac{p_i}{A} - \lfloor \frac{p_i}{A} \rfloor$ (the fractional part of the quotas)
3. Distribute the remaining seats to the states with the highest remainders.

Example

Let $\mathbf{p} = (630, 480, 390, 500)$ and $H = 10$.

	Population	Share ($\frac{p_1}{A}$)	$\lfloor \frac{p_i}{A} \rfloor$	Hamilton
State A	630	3.15	3	3
State B	480	2.40	2	2
State C	390	1.95	1	2
State D	500	2.50	2	3
Total	$P = 2000$	$A = 200$	8	10

The Hamilton method satisfies the Hare Quota Property by definition.

Divisor methods

Alabama-paradox

During the 1880 US census the Chief Clerk of the Census Office considered an enlargement of the House of Representatives and noted that moving from 299 to 300 seats would result in a loss of a seat for the State Alabama.

Alabama-paradox

A state gets less seats when the size of the House is increased (while every other parameter, including the population remains fixed).

House-monotonicity

An apportionment method M is **house-monotonic** if $M(\mathbf{p}, H')_i \geq M(\mathbf{p}, H)_i$ for all $H' \geq H$ and $i \in N$: if no state loses a seat by increasing the House size.

House-monotonicity is synonymous w/ resource-monotonicity (cake cutting).

Divisor methods – D'Hondt method

D'Hondt method

1. Distribute 1 (or 0) seat to every state.
2. Allocate a seat to the state with the largest $\frac{p_i}{a_i+1}$, $i = 1, 2, \ldots, n$ ratio.
3. Continue the process until $\sum_i a_i$ becomes equal to H (that is until all the seats have been distributed).

Example

Let $\mathbf{p} = (630, 480, 290)$ and $H = 7$.

State	Share	Seats
State A	$630/2 = 315$	1
State B	$480/2 = 240$	1
State C	$290/2 = 145$	1
Total		3

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State	Share	Seats
State A	$630/3 = 210$	2
State B	$480/2 = 240$	1
State C	$290/2 = 145$	1
Total		4

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State	Share	Seats
State A	$630/4 = 157.5$	3
State B	$480/3 = 160$	2
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Total		6

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Remarks

- Thomas Jefferson, founding father of the USA proposed an equivalent method (although procedurally they are different): the two methods always yield the same result. In the US the D'Hondt method is referred as Jefferson method.
- Apportionment methods can be applied to party lists: seats are distributed among parties based on the votes they receive. D'Hondt method is used e.g. in Hungary for party lists.

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Notable divisor methods

Adams method	(current value)	$\frac{p_i}{a_i}$
Dean method	(harmonic mean)	$\frac{p_i}{\frac{2}{\frac{1}{a_i}+\frac{1}{a_i+1}}}$
Huntington-Hill method/EP	(geometric mean)	$\frac{p_i}{\sqrt{a_i(a_i+1)}}$
Sainte-Laguë/Webster method	(arithmetic mean)	$\frac{p_i}{a_i+1/2}$
Jefferson/D'Hondt method	(next value)	$\frac{p_i}{a_i+1}$

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State	Population	Adams	Dean	EP	Webster	D'Hondt
A	9061	9	9	9	9	10
B	7179	7	7	7	8	7
C	5259	5	5	6	5	5
D	3319	3	4	3	3	3
E	1182	2	1	1	1	1
Total	26000	26	26	26	26	26

Exercise

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We want to distribute 8 fellowship positions among three programs based on the number of enrolled student, which are respectively 476, 316 and 208. Distribute the positions with the Adams, D'Hondt and the Hamilton methods!

Hint: The numbers add up to 1000.

divided by:	1	2	3	4	5	6	7	8
476	476	238	158.7	119	95.2	79.3	68.0	59.5
316	316	158	105.3	79	63.2	52.7	45.1	39.5
208	208	104	69.3	52	41.6	34.7	29.7	26.0

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Adams (4, 3, 1), D'Hondt (5, 2, 1), Hamilton (4, 2, 2)

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Example

Distribute 50 seats with the Hamilton method among five states.

State	Population	Quota	Apportionment
A	78	4.33	4
B	150	8.33	8
C	173	9.61	9
D	204	11.33	11
E	295	16.39	16
Total	900		48

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A	78→ 78	4.33→ 4.29	4→
B	150→ 150	8.33→ 8.25	8→
C	173→ 181	9.61→ 9.96	10→
D	204→ 204	11.33→11.22	11→
E	295→ 296	16.39→16.28	17→
Total	900→ 909	→	50→

Population paradox

Between 1900 and 1910 the population of State Virginia grew more rapidly than that of Maine, yet it lost a seat in the House while Maine gained one!

Population paradox

Population paradox occurs when a state with a more dynamic population growth loses a seat to a state with less growth (ceteris paribus).

Example

Distribute 50 seats with the Hamilton method among five states. Suppose that, the populations of states C and E grows.

State	Population	Quota	Apportionment	
A	78→ 78	4.33→ 4.29	4→ 4	1
B	150→ 150	8.33→ 8.25	8→ 8	
C	173→ 181	9.61→ 9.96	10→ 9	1
D	204→ 204	11.33→11.22	11→11	
E	295→ 296	16.39→16.28	17→16	
Total	900→ 909	→	50→48	

State E loses a seat to A while A did not grow at all!

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A	78 → 78	4.33 → 4.29	4 → 5
B	150 → 150	8.33 → 8.25	8 → 8
C	173 → 181	9.61 → 9.96	10 → 10
D	204 → 204	11.33 → 11.22	11 → 11
E	295 → 296	16.39 → 16.28	17 → 16
Total	900 → 909	→	50 → 50

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Total	900→ 909	→	50→ 50

State E loses a seat to A while A did not grow at all!

New state paradox

In 1907, Oklahoma joined the Union and the House was enlarged with 5 seats, which was the fair share of Oklahoma. By re-applying the Hamilton method, New York lost a seat to Maine although neither states' population changed.

New state paradox

This paradox occurs when a new state joins to the existing ones and although the House is enlarged with the number of seats that the new state gets another state receives less seats than before.

Example

In the following example we apply the Hamilton method.

State	Population	Share	Result
A	1 045	10.450	10
B	8 955	89.550	90
???	???	???	???
Total	10 000		100

New state paradox

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New state paradox

This paradox occurs when a new state joins to the existing ones and although the House is enlarged with the number of seats that the new state gets another state receives less seats than before.

Example

In the following example we apply the Hamilton method.

State	Population	Share	Result
A	1 045	10.425	11
B	8 955	89.337	89
C	525	5.237	5
Total	10 525		105

Remarks

- Divisor methods are House- and population-monotonic and immune to the new-state paradox.
- Divisor methods do not satisfy the Hare quota property.
- No method is House- and population-monotonic and satisfies Hare quota (Balinski and Young, 1975)

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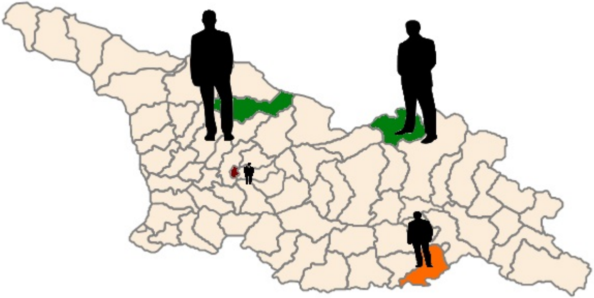
  - Quota method (Balinski and Young, 1975) is Quota and free of the Alabama paradox, but exhibits the population paradox
- The seats of the US House of Representatives is distributed among states using the EP method among the states.

Part II

The Venice Commission and the Maximum Admissible Difference

An infamous example



The 1999 Electoral law of Georgia did not set rules about the sizes of constituencies. The number of voters per (single-seat) constituencies ranged from 3 600 in the Lent'ekhi or 4 200 in the Kazbegi districts to over 138 000 in Kutaisi City, hugely favouring voters in the former regions.
Such differences question the validity of the whole election.

Venice Commission

European Commission for Democracy through Law (Est: 1990)

- *To aid the codification of the constitution and electoral law of newly founded democracies.*
- Published a guidebook for electoral laws: The Code of Good Practice in Electoral Matters (Venice Commission, 2002)
- EU observers consistently refer to this book when reviewing elections and legal processes (e.g. 2011 Albanian and Estonian electoral laws).

The Venice Commission's recommendation

"***Equality in voting power**, where the elections are not being held in one single constituency, requires constituency boundaries to be drawn in such a way that seats in the lower chambers representing the people are distributed equally among the constituencies, in accordance with a specific apportionment criterion ...*"

"*Constituency boundaries may also be determined on the basis of geographical criteria and the **administrative or** indeed **historic boundary lines**s, which often depend on geography ...*"

"*The **maximum admissible departure from the distribution criterion adopted** depends on the individual situation, although it **should seldom exceed 10% and never 15%**, except in really exceptional circumstances (a demographically weak administrative unit of the same importance as others with at least one lower-chamber representative, or concentration of a specific national minority).*"

The Code of Good Practice in Electoral Matters (2002), Section 2.2 §13–15

Maximum admissible departure

Distribution criterion adopted?

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Definitions

- Let δ_i be the relative difference from the average of the typical (average) constituency of state i, and
- let d_i be its absolute value, the *departure*.

$$\delta_i = \frac{\frac{p_i}{a_i} - A}{A} \quad \text{and} \quad d_i = |\delta_i|.$$

$\therefore$ Venice Commission: $\max_{i \in N} d_i \leq 10\%$ (or 15% in the worst case).

Hare quota vs. VC's recommendation

The Hare quota and the Venice Commission's recommendation is similar in spirit but there is a fundamental difference (Kóczy, Biró, and Sziklai, 2017).

- Hare quota specifies how many seats a ***state*** should get at least or at most. If the allocation violates Hare quota it is unfair from the point of view of the particular state.
- VC's recommendation focuses on the equality of the ***individual voters***. If the sizes of constituencies differ too much so do the voters' abilities to influence decisions.

Maximal difference property vs Hare quota

Example

Solve $(20, (26, 27, 28, 29, 91))$.

		Method $\Rightarrow$	Maximal difference			Hare-quota		
State	Popl'n	quota	seats	$\frac{p_i}{a_i}$	δ_i	seats	$\frac{p_i}{a_i}$	δ_i
A	26	2.59	3	8.67	$-\mathbf{0.14}$	2	13.00	**0.29**
B	27	2.69	3	9.00	-0.10	3	9.00	-0.10
C	28	2.79	3	9.33	-0.07	3	9.33	-0.07
D	29	3.79	3	9.67	-0.04	3	9.67	-0.04
E	91	9.05	8	11.38	0.13	9	10.11	0.01
Total	201	20	20	10.05		20	10.05	

If we insist on Hare-quota, State E must get at least 9 seats.

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Total	201	20	20	10.05		20	10.05	

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Total	201	20	20	10.05		20	10.05	

If we insist on Hare-quota, State E must get at least 9 seats.
$\therefore$ VC's condition is not satisfied.
Conflict between Hare Quota and VC's recommendation. How to formalise it?

Maximal difference property

The maximal difference property is not compatible with house-monotonicity either. An apportionment rule that minimizes the maximal difference can produce the Alabama-paradox.

Example

Consider the problem (14, (69, 70, 150), then increase House size to 15.

State	Popl'n	Seats	$\frac{p_i}{a_i}$	d_i	Seats	$\frac{p_i}{a_i}$	d_i
A	69	3	23.00	0.11	4	17.25	-0.10
B	70	3	23.33	0.13	4	17.50	-0.09
C	150	8	18.75	-0.09	7	21.43	0.11
Total	289	14	20.64		15	19.27	

Maximum Admissible Departure

Note: for a given House size h it may be *mathematically impossible* to find an allocation where $\max_{i \in N} d_i \leq 10\%$ (or even 15%).

We look for an allocation that *minimizes* the maximum departure.

For an apportionment problem $(\mathbf{p}, H)$ let $\alpha(\mathbf{p}, H)$ denote the smallest maximum departure achievable, that is

$$\alpha(\mathbf{p}, H) = \min_{\mathbf{a}} \max_{i \in N} \{d_i\}$$

Maximum Admissible Departure (MAD) Property

An apportionment method M satisfies the *MAD* Property if for all $(\mathbf{p}, H)$,

$$\max_{i \in N} \left| \frac{\frac{p_i}{M(\mathbf{p},H)_i} - A}{A} \right| \leq \alpha(\mathbf{p}, H).$$

Maximum Admissible Departure and House size

Note: α is not monotone in the House size.

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Example

For instance let $p_1 = 100$, $p_2 = 200$ and let $H = 3$. Then it is possible to distribute the seats according to the exact quota thus $\alpha = 0$.
Increasing H by 1 however will spoil both d_1 and d_2.

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By large, however a higher H gives a lower α.
The problem with small House sizes is that they imply a larger average constituency size. Divisibility issues can appear for smaller states that are only a few times as large as A. In the worst case the average size of the constituencies of state i is equally far away from A for both the lower and upper integer part of $\frac{p_i}{P}H$, formally

$$\frac{\frac{p_i}{l_i} - A}{A} = \frac{A - \frac{p_i}{u_i}}{A}.$$

Maximum Admissible Difference and small states

$$\frac{\frac{p_i}{l_i} - A}{A} = \frac{A - \frac{p_i}{u_i}}{A}.$$

Example

If $l_i = 2$ and $u_i = 3$ then $p_i = \frac{12}{5}A$ and $d_i = \frac{1}{5} = 0.2$.

Maximum Admissible Difference and small states

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Example

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In general, $d_i = \frac{1}{2l_i+1}$:

$l_i - u_i$	p_i	β_i
$1-2$	$\frac{4}{3}A$	$0.33=\frac{1}{3}$
$2-3$	$\frac{12}{5}A$	$0.20=\frac{1}{5}$
$3-4$	$\frac{24}{7}A$	$0.14=\frac{1}{7}$
$4-5$	$\frac{40}{9}A$	$0.11=\frac{1}{9}$
$5-6$	$\frac{60}{11}A$	$0.09=\frac{1}{11}$

Table: Critical state population regarding divisibility

Maximum Admissible Difference and small states

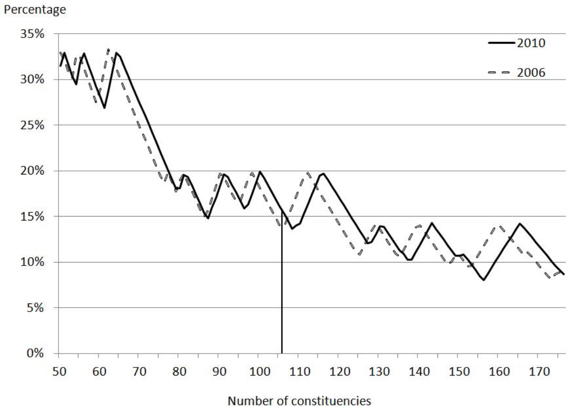

Figure: The decline of maximal difference compared to increasing House size using voter data from 2006 and 2010.

Maximum Admissible Difference: Summary

- Venice Commission: a fundamentally new approach focusing on individual voters
- Incompatible with former models and even properties
- ía

Optimization methods

The Leximin method

We want a method that

- Conforms to the Venice Commission's recommendation
- Goes further: minimizes the maximum departure(?)
- Is unique.

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The **Leximin method** (Biró, Kóczy, and Sziklai, 2015) *lexicographically minimizes departure*, that is,

(1) first it minimizes the maximum departure,

(2) among allocations that satisfy (1) find one that minimizes the second largest departure,

(3) among allocations that satisfy (2) find one that minimizes the third largest departure,

(...) and so on until we find a (generically) unique solution.

The Leximin method: Example

Example

Allocation	*departure (%)*		
1	16	26	19
2	24	13	19
3	24	26	7

The Leximin method: Example

Example

Allocation	*departure (%)*		
1	16	26	19
2	24	13	19
3	24	26	7

Allocation	Worst	2nd worst	3rd worst
1	26	19	16
2	24	19	13
3	26	24	7

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Lexicographic sorting: “like sorting alphabetically”

The Leximin method: Example

Example

Allocation	*departure (%)*		
1	16	26	19
2	24	13	19
3	24	26	7

Allocation	Worst	2nd worst	3rd worst
1	A	I	L
2	G	I	N
3	A	G	O

Lexicographic sorting: “like sorting alphabetically”

Value	26	24	19	16	13	7
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Example Let $\mathbf{a} = (3, 14, 1, 8)$, $\mathbf{b} = (3, 6, 20, 30)$, $\mathbf{c} = (3, 14, 2, 1)$.
Then **b** is smallest vector in lexicographic sense, as the first component of **b** is the same as for **a** and **c**, but its second component is smaller than the second component of those two.
c is the largest in lexicographic sense, as its first two component is the same as for **a**, but its third component is larger.

The Leximin method: Exercise

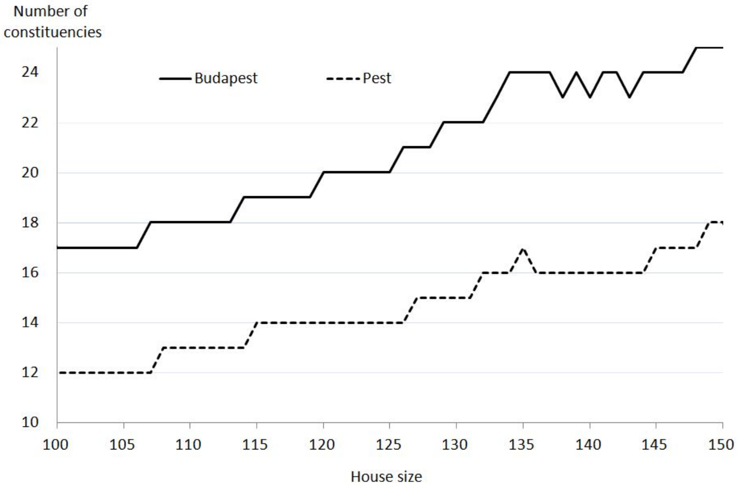
A university has three programmes (A, B, C), the number of enrolled students are $p_A = 60$, $p_B = 93$, $p_C = 63$ respectively. The university wants to put together a committee that represents all of the three programmes. Suppose we would like an allocation that is conform with the VC's recommendation. How many delegates should each programme send? To make matters simple we only consider the following three allocations: (3,3,3), (2,4,3), (2,5,2).

- Which allocation does not satisfy the MD-property?
- Which allocation is lexicographically minimal?

The Leximin method: Properties

The Leximin method, and every method which is conform with the Venice Commission's recommendation violates Hare Quota and is neither House- nor Population-Monotonic.

The Leximin method: Violates House-monotonicity



Occurrences of the Alabama-paradox in case of Budapest and Pest county using Leximin method

Hare-Quota violations

To asses how often a method violates the Hare quota property we look at the US House of Representatives for House sizes 335 to 535 (actual size $\pm$100).

	Leximin	EP	D'Hondt	Adams	Webster
California	112	2	201	201	0
Texas	30	0	198	192	0
New York	12	0	120	67	0
Florida	6	0	105	21	0
Other	0	0	19	24	0

Leximaximin and Leximinimax methods

Leximinimax method

- Sort average district sizes into decreasing order.
- Lexicographically minimise maximum.

Leximaximin method

- Sort average district sizes into increasing order.
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Motivation?

Leximaximin and Leximinimax methods

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Motivation? Efficient use of resources

- Leximaximin: allocating human resources while maximising service quality (post office workers, quality assurance)
- Leximinimax: least wasteful use of available resources (allocating grants, combating crime)

Leximaximin/Leximinimax: Properties

Theorem

The Leximaximin method and the Adams method coincide.

Theorem

The Leximinimax method and the D'Hondt/Jefferson method coincide.

Corollary

The Leximin method lies between the D'Hondt/Jefferson and Adams methods.

(This is the first result to link the Leximin Method to historic apportionment methods.)

Apportionment methods: a comparison

	Hare quota	house mon.	pop. mon.	Venice Commission
Hamilton	Yes	No	No	No
Jefferson/D'Hondt	Lower	Yes	Yes	No
Webster/Sainte-Laguë	Mostly	Yes	Yes	No
Huntingdon-Hill/EP	Mostly	Yes	Yes	No
Adams	Upper	Yes	Yes	No
Leximin	No	No	No	Yes

Malapportionment (Kóczy and Sziklai, 2018)

Violating Hare quota

Example

100 seats are distributed among 5 states. The avg district size, $A = 110$. (l_i and u_i are i's quota rounded down and up; $d_i(x)$ is the relative difference from A when the state receives x seats.)

	Population	Share	l_i	u_i	$d_i(l_i)$	$d_i(u_i)$
State A	10000	90.90	90	91	1.0%	0.1%
State B	252	2.29	2	3	14.5%	23.6%
State C	251	2.28	2	3	14.0%	23.9%
State D	249	2.26	2	3	13.1%	24.5%
State E	248	2.25	2	3	12.7%	24.8%
Total	11000	100	98	102		

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Optimally A gets 91, B-E get 2 each.Who gets the last seat?
State B? d_B and d badly increased. (C-E even worse.)
State A? d_A hardly increased, d does not change.

Hare quota and departure

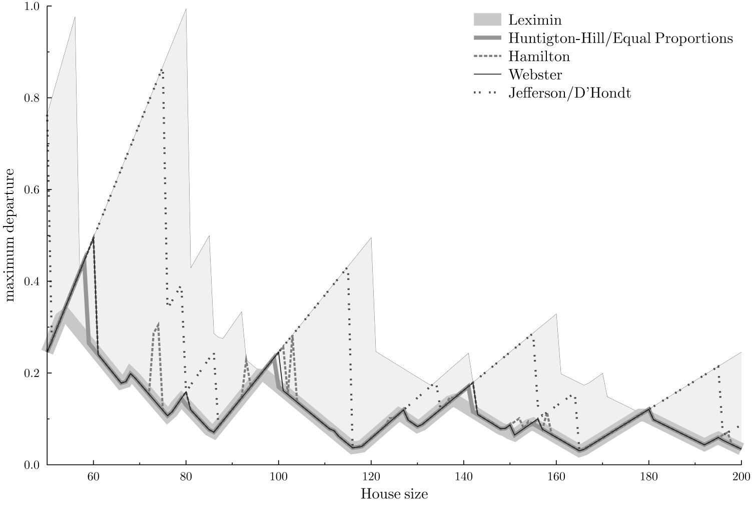
The difference from average constituency size is the smallest if *i* receives either its lower (l_i) or upper quota (u_i), although it matters which one.
In the best case scenario, we can round the shares of the critical states in the right direction, we can achieve β difference from the average.
In the worst case, scenario, when round the critical states in the wrong direction, we obtain ω difference.

$$\beta_i = \min\left(\left|\frac{\frac{p_i}{l_i} - A}{A}\right|, \left|\frac{\frac{p_i}{u_i} - A}{A}\right|\right), \qquad \beta = \max_{i \in N} \beta_i.$$
$$\omega_i = \max\left(\left|\frac{\frac{p_i}{l_i} - A}{A}\right|, \left|\frac{\frac{p_i}{u_i} - A}{A}\right|\right), \qquad \omega = \max_{i \in N} \omega_i.$$

The maximum departure of a method that satisfies Hare quota is always between β and ω.

Hare quota and departure: Example (Belgian data)

The grey area represents the interval between β and ω.



Hare quota and departure

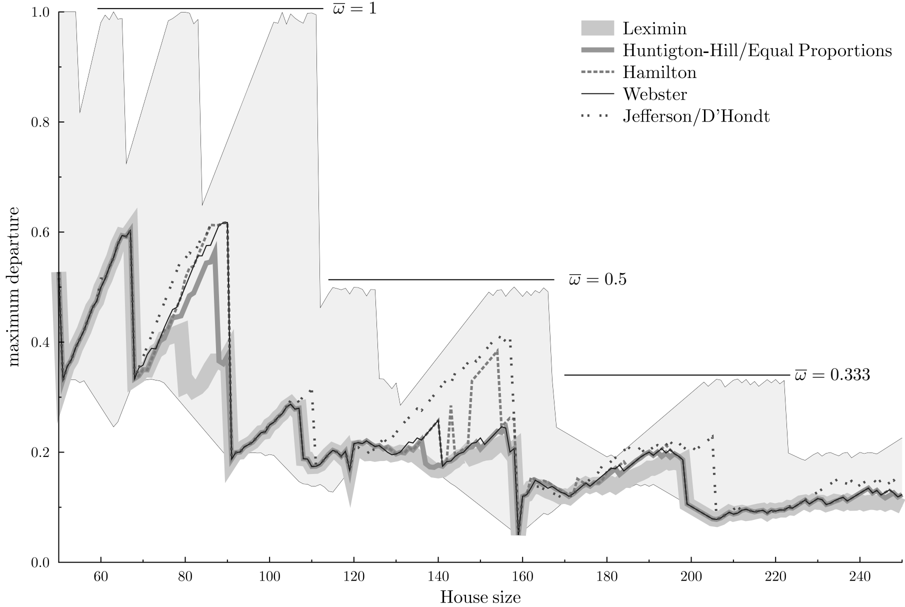
Observe: the largest difference between rounding in the right or wrong direction may emerge for the smallest states. This has an effect on ω:

Theorem

Let l_{sm} denote the lower integer part of the respective share of the smallest State. Then an upper bound for ω, can be given as

$$\omega \leq \overline{\omega} \stackrel{def}{=} \begin{cases} \frac{1}{l_{sm}} & \text{if } l_{sm} > 0, \\ \infty & \text{if } l_{sm} = 0. \end{cases}$$

Departure with respect to House size (Irish data)



New Hungarian Electoral Law

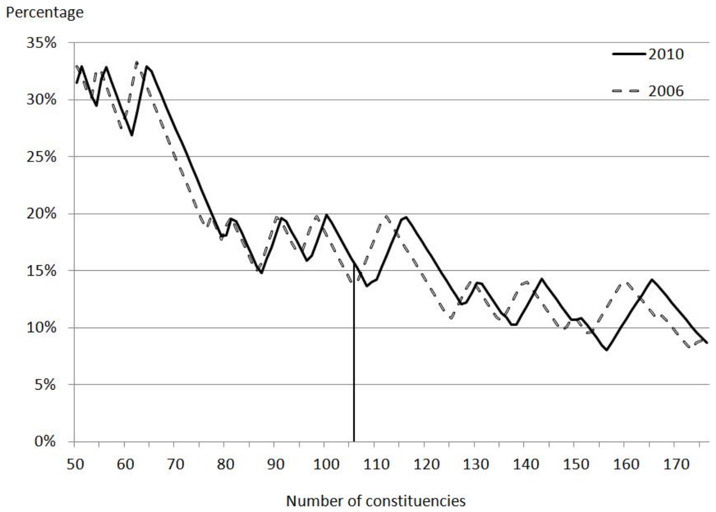
- In 2011 Hungary introduced a new electoral law in the spirit of the Venice Commission's recommendation. It states that the difference between the constituency size of any county and the average constituency size cannot be larger than 20%.
- The number of seats allocated by the law is very close to the Leximin (but not identical, and no procedure is given how to calculate it).
- The largest difference from the average (d) is produced by the constituencies of Tolna county with only 65 583 voters ($A = 77\,415$), which is 15.2% smaller.
- Largest districts are in Csongrád with 86 486 voters.
- The difference between the counties of Tolna and Csongrád is 31.8%. That is, the voters of Tolna are 31.8% more influential than those of Csongrád.

Decreasing the gap

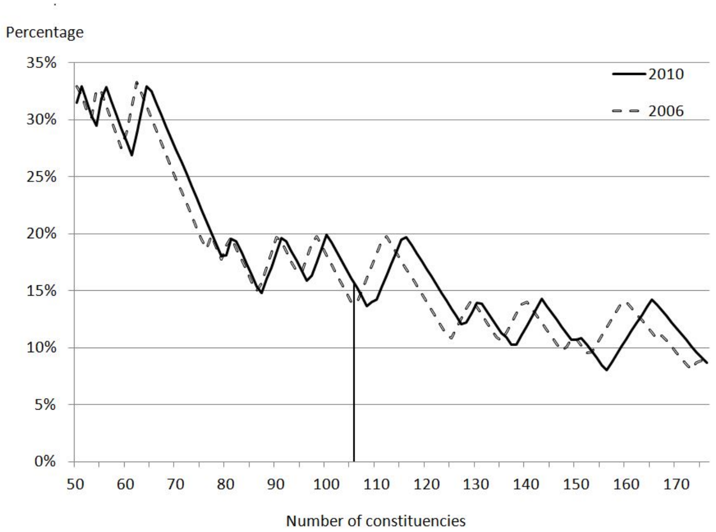
As a result of any apportionment, some voters are more influential than others. But large differences question the validity of the whole election. By lowering the maximum departure, we can decrease the gap between the voters influences. How can we do that?

- Use Leximin, or any method that minimizes the maximum departure.
- Increase the House size.
- Aggregate States/Counties into larger regions.

Departure wrt House size (Hungarian data)

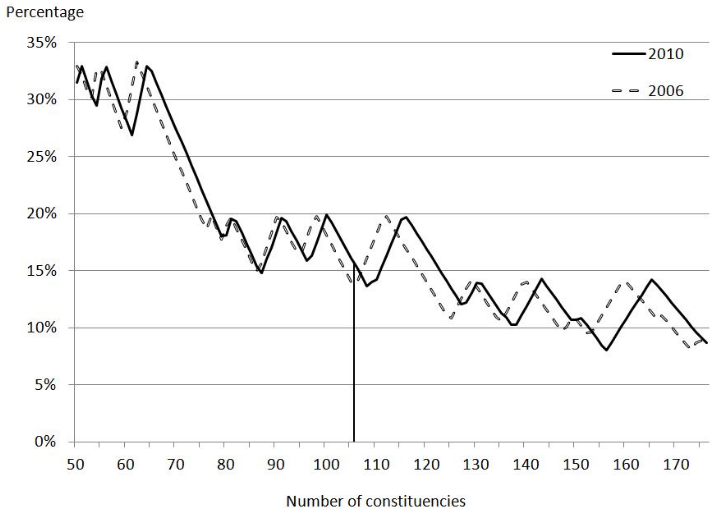


Departure wrt House size (Hungarian data)



106 is optimal wrt 2006 data

Departure wrt House size (Hungarian data)



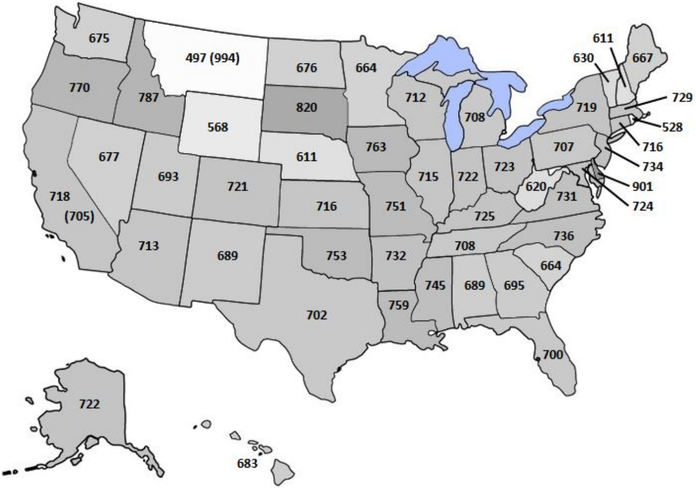
106 is optimal wrt 2006 data but over time departure can go over 20%*.

Aggregating counties into larger regions

If we aggregate the 20 counties of Hungary into 7 regions, then the departure drops drastically, especially if we use the Leximin method.

Region	#Voters	#Seats		Departure (%)	
		by law	Leximin	by law	Leximin
North-Hungary	995 863	12	13	10.87	1.05
Northern Great Plain	1 215 043	16	16	5.35	1.90
Southern Great Plain	1 092 768	14	14	11.72	0.83
Central-Hungary	2 381 138	30	30	4.81	2.53
Central-Transdanubia	906 714	12	12	9.97	2.40
West-Transdanubia	822 903	11	11	7.09	3.37
South-Transdanubia	791 538	11	10	15.28	2.25
Total	8 205 967	106	106		

Constituency sizes in the USA



Contituency sizes in the USA (1000 persons, for Leximin (EP))

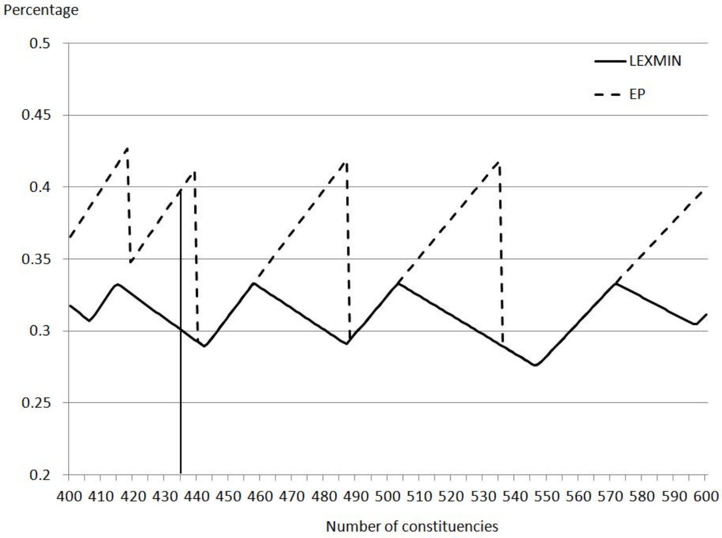
Constituency sizes in the USA

- Under EP, Montana has the largest constituencies with 994k voters, the smallest are in Rhode Island with 528k voters. The constituencies of Montana are almost twice as large as the ones in Rhode Island. Assuming that the voters' influence is proportional to the size of the constituencies, the voters of Rhode Island have 88% more influence than the voters of Montana.

Constituency sizes in the USA

- Under EP, Montana has the largest constituencies with 994k voters, the smallest are in Rhode Island with 528k voters. The constituencies of Montana are almost twice as large as the ones in Rhode Island. Assuming that the voters' influence is proportional to the size of the constituencies, the voters of Rhode Island have 88% more influence than the voters of Montana.
- In contrast, within one state virtually no deviation is allowed. The US Supreme court forced Missouri to redesign its constituencies because it found that the 3.3% difference between them was too large (Kirkpatrick v. Preisler,1969)!
- Leximin gives one more seat to Montana (and one less to California), which decreases this gap somewhat (but it's still too large).
- To guarantee that the difference from the average constituency size does not exceed 20%, the size of the House should be doubled.

Maximum departure in the USA



Maximum departure vs House size in the House of Representatives

Conclusion

How to decrease departure?

- Use a method that minimizes the maximum departure
- Increase the House size
- Choose the House size from an interval
- Aggregate states/counties into larger regions

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