Game-Theoretic Models of Partnership Formation

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Introduction

In the economics literature, such problems are called *job search problems*.

In the biological literature, such problems are called *mate search problems*.

Overview

- 1. The classical mate choice problem.
- 2. Game theoretic aspects of mate choice.
- 3. Practical aspects of mate choice.
- 4. Game-theoretic models when only "females" are choosy
- 5. Game-theoretic models when both sexes are choosy.
- 6. Some obvious omissions and directions for future research.

The Classical Mate Choice Problem (see Janetos [1980], Real [1990, 1991])

- It is assumed that only females are choosy.
- Females observe a sequence of prospective partners (males), whose values are i.i.d. realizations of a non-negative random variable, X.
- It is assumed that the distribution of X is implicitly known (via evolution).
- On observing a male, a female can either a) reject him and carry on searching, or b) accept him, and thus stop searching.
- Search costs are assumed to be linear in the number of males observed, (i.e. c per male).

The Classical Mate Choice Problem

- The reward obtained by a female is assumed to be the value of the male accepted minus the search costs, i.e. $X_T - cT$, where T is the number of males seen.
- a) Assume that the female can (at least in principle) observe an infinite number of males.
- The optimal strategy is a fixed threshold, which is equal to the expected reward from search.
- Hence, at each stage a female accepts a male if and only if his value is greater than the expected reward from future search (ignoring previously incurred costs).

The Classical Mate Choice Problem

Let the optimal threshold be x^*. The optimality equation is given by

$$x^* = \int_0^{x^*} x^* f(x)dx + \int_{x^*}^{\infty} xf(x)dx - c.$$

This can be rearranged to

$$c = \int_{x^*}^{\infty} (x - x^*)f(x)dx.$$

As search costs increase, females are less choosy.

Heavy-tailed distributions of mate value tend to lead to choosier females.

The Classical Mate Choice Problem

- b) Suppose that the maximum number of males a female can observe is N.
- In this case, the optimal strategy is given by a set of decreasing thresholds (which can be determined recursively via dynamic programming).
- Again, a female accepts a male if and only if his value is greater than the expected reward from future search.
- **Note**: Similar results can be obtained when males appear according to a Poisson process.
- When there is a maximum search time, the optimal threshold is a decreasing function of time.

Some game theoretic aspects of mate choice problems

- One-sided mate choice vs. two-sided mate choice.
- Preferences (common and/or homotypic).
- Steady-state dynamics vs. scramble competition.
- Changing characteristics of searchers (e.g. age).

Some game theoretic aspects of mate choice problems

Leks

Some game theoretic aspects of mate choice problems

Common and Homotypic preferences

Some game theoretic aspects of mate choice problems

Age/ Resource Holding Potential

Game Theory and One-Sided Mate Choice - Scramble Competition

Collins and McNamara [1993] consider a model of scramble competition with one-sided choice.

Mathematically speaking, this is formulated as a mate choice problem, but in biological terms it might be better interpreted as, e.g. search for a nesting site.

Consider a large (essentially infinite) population of birds and a large population of nesting sites (such that there are at least as many nesting sites as birds).

The value of the nesting sites follow a known distribution.

Game Theory and One-Sided Mate Choice - Scramble Competition

As the season progresses, the most attractive nesting sites are occupied. Hence, the distribution of the values of available nesting sites changes over time.

Although choice is one-sided, this model is naturally game theoretic, since the optimal strategy of a single searcher depends on the strategies used by other searchers (which affect the distribution of the nesting sites available at each moment.

The authors derive the form of the ESS for this problem (it is a threshold strategy).

Even when the available search time is essentially unlimited, this threshold is decreasing in time, since the best sites are occupied first.

Game Theory and One-Sided Mate Choice - Scramble Competition

Dechaume-Moncharmont [2016] give numerical results for a mate-choice problem for a seasonal breeder in which only females are choosy.

Males and females only breed once each season.

Females find prospective partners according to a Poisson process.

Individuals can enter the mating pool at different times at the beginning of the mating season.

Females become less choosy as time progresses.

Game Theory and One-Sided Mate Choice - Latency

Even when only females are choosy and males do not provide any support post-mating, mate choice problems are game-theoretic in nature, due to the fact that males can have latent periods.

This means that the distribution of the values of available males depends on the strategies used by females.

Etienne *et al.* [2014] consider such a steady-state model of mate choice and derive the evolutionarily stable strategy (ESS) of females.

Game Theory and One-Sided Mate Choice - Latency

When a mate-choice problem is not focused on a single decision maker, then it is necessary to define the general nature of preferences.

Here, females are assumed to have common preferences.

Game Theory and One-Sided Mate Choice - Leks

Hutchinson and Halupka [2004] consider a model of one-sided mate choice in which males are not necessarily distributed uniformly in space.

At one end of the spectrum, males are distributed evenly in space.

In this case, from the point of view of a single female the classical model seems reasonable (search costs are proportional to the number of males seen).

At the opposite end of this spectrum, males may gather together in groups, called leks.

Game Theory and One-Sided Mate Choice - Leks

When leks are of reasonably large size, it is optimal for females to inspect all the males in a lek and choose the most attractive.

In the real world, the larger the size of a lek, the greater the number of females visiting a lek.

Females are more likely to pair with "high ranking" males, but differ in their choice.

Game Theory and One-Sided Mate Choice - Leks

Kokko [1997] and Hernandez *et al.* [1999] investigate these issues via a game-theoretic model.

As the size of a lek increases, then each male (regardless of rank) obtains a lower proportion of pairings with females.

However, when the lek size is small this is compensated by a higher encounter rates with females.

Males thus form leks of intermediate size. However, there may be some conflict between males regarding the size of leks.

Game Theory and Two-Sided Mate Choice

Game Theory and Two-Sided Mate Choice - Static Models

In the models considered here, it is assumed that the adult sex ratio is equal to 1.

It is also assumed that pairs are formed only by mutual acceptance.

Two-sided mate choice is by definition game theoretic.

McNamara and Collins [1990], as well as Parker [1983] consider static models of two-sided mate choice with common preferences.

The distributions of the values of mates according to sex are assumed to be fixed.

Game Theory and Two-Sided Mate Choice - Static Models

Since searchers are assumed to leave the mating pool after finding a partner, static models can also be interpreted as "clone-replacement models".

When a pair leave the mating pool, they are replaced by two searchers with identical characteristics.

According to the model presented by McNamara and Collins [1990], searchers of both sexes observe a sequence of prospective partners (similar to the classical mate search problem).

The time available for search is unlimited.

Static Models

Consider the problem faced by a female of maximum attractiveness.

Since she should be accepted by any male, she faces a one-sided search problem.

Hence, such females should only accept males of value, say, $\geqslant x_1$.

Analogously, males of values $\geqslant x_1$ face a one-sided problem and should only pair with females of value, say, $\geqslant y_1$.

Together, these males and females may be called type 1 and only accept prospective partners of type 1.

Static Models

Since the remaining searchers are always rejected by type 1 searchers, we can reduce the game faced by the other players by removing type 1 players.

Then using an analogous argument, we can define type 2 males and females, who pair exclusively with each other.

Calculating inductively both sexes can be split into k types, such that type i males pair only with type i females.

The value of k depends on the search costs and the distributions of the values of mates.

There may be a group of males or females who never pair.

Static Models

Ramsey [2012] considers a model in which searchers exhibit both common and homotypic preferences.

Each searcher prefers attractive partners (females agree as to how attractive a male is), who are similar in character to themselves.

It is assumed that attractiveness can be observed quickly, but in order to observe the character of a prospective partner, dating is required.

Equilibria are derived for such games.

Steady-State Models

State-state models assume that males and females enter the mating pool at a constant rate and the distribution of the values of these entrants is fixed

(i.e. is not by definition identical to the distributions of the values of males and females leaving the mating pool).

Burdett and Coles [1999] consider such a model of job search (males and females are replaced by job-seekers and positions).

Steady-State Models

Given that the population are using a given set of strategies, the distribution of the values of positions and the values of job-seekers tend to a steady-state distribution.

Hence, the optimal response of a player is to use a threshold strategy.

Thus any equilibrium is defined by a set of thresholds.

However, derivation of an equilibrium is, in general, difficult and games can have multiple equilibria.

Game Theory and Two-Sided Mate Choice

Burdett and Coles consider a game of this form in which there are two types of job-seeker (good and bad) and two types of position (good and bad).

When both good job-seekers and good positions are rare (a proportion p of job-seekers and positions alike), there can be two equilibria.

Game Theory and Two-Sided Mate Choice - Steady-State Models

Non-choosy: Each job-seeker occupies the first position found.

Good positions are rare in the pool of available jobs (a proportion *p*) and it does not pay a good job-seeker to wait for a good position.

Choosy: Good job-seekers only accept good positions (mutually). Bad job-seekers accept any type of position, but are only offered bad positions.

Good job-seekers spend more time looking for a job than bad job-seekers, but this means that the proportion of good positions in the pool of available jobs is significantly greater than *p*.

Hence, it pays for good job-seekers to be choosy.

A Steady-State Model with Preferences Based on Age

Naturally, the characteristics of an individual will change during his/her life.

The most obvious trait of this form is age.

Alpern *et al.* [2013] consider a game in which males and females enter the population at a steady rate (at age 0).

Each player searchers for a partner until either they find one or lose their fertility (at age 1).

A Steady-State Model with Preferences Based on Age

Prospective partners are found according to a Poisson process.

The age of the prospective partner comes from the distribution of the ages of individuals in the mating pool (which depends on the strategies used).

The payoff obtained by a pair is a function of their ages.

In the simplest model, this payoff is assumed to be the time until the eldest member of the pair loses his/her fertility.

A Steady-State Model with Preferences Based on Age

Under fairly natural conditions, it can be shown that the equilibrium strategy is given by a threshold function.

A searcher of age x accepts a prospective partner of age $\leqslant T(x)$, where $T(x)$ is increasing in x and $T(x) \geqslant x$.

A policy iteration algorithm is used to derive the equilibrium strategies.

A Steady-State Model with Preferences Based on Age

In iteration 1, it is assumed that females do not pair and males always accept females.

The optimal threshold strategy of an individual female, $T_1(x)$ is calculated using a numerical procedure, along with the probability that such a female is still searching at age x, $p_1(x)$.

The optimal strategy of an individual male when females use the threshold rule T_1 and the distribution of their ages corresponds to p_1 is derived in a similar way.

This procedure is repeated until convergence.

A Steady-State Model with Preferences Based on Age

Ramsey [2012] extended the results of these models to include

- Mortality.
- Various rules for determining the rates of interaction.
- Games where the adult sex ratio is not one (males and females enter the mating pool at different rates and/or are fertile for different lengths of time).

Models of Scramble Competition

Models of Scramble Competition

Models of scramble competition assume that individuals begin looking for a mate at the beginning of the mating season.

Once a pair is formed, the individuals involved leave the pool of searchers.

Hence, the distribution of the values of prospective partners in the mating pool changes over time according to the strategies used in the population as a whole.

Models of Scramble Competition - Discrete Time

Johnstone [1997] considers a model of scramble competition with n rounds and common preferences.

In each round, each searcher is paired with a prospective partner from the current mating pool.

If such a pair is mutually acceptable, then they form a pair and leave the mating pool.

The distribution of the values of prospective partners (assumed to be independent of sex) is a discretized version of the normal distribution.

Models of Scramble Competition - Discrete Time

Solutions are found by policy iteration.

The best response of each searcher to the current set of threshold strategies (and resulting distributions of mate values at each moment) is calculated via dynamic programming

This best response is then used as the current set of threshold strategies in the next iteration.

The iterations are continued until the process converges.

Intuitively, all searchers accept any prospective partner in the n-th round (the last prospective partner).

Models of Scramble Competition - Discrete Time

Generally, searchers become less choosy as the mating season progresses.

However, searchers of low attractiveness may show an increase in choosiness towards the end of the mating season (but before the last round).

This is due to the chance of pairing with an attractive partner in the final round.

Models of Scramble Competition - Discrete Time

Alpern and Reyniers (2005), Alpern and Katrantzi (2009), as well as Mazolov and Falko (2008), derive more analytic results for games of this form.

Alpern and Reyniers (1999) consider a model where preferences are homotypic.

Initially the distribution of the "character" of searchers is uniform (independently of sex).

In such games, individuals of "central character" are more choosy.

Models of Scramble Competition - Continuous Time

Ramsey [2015], [2018] (+ current work) considers models of scramble competition when time is continuous and the population size is large.

In the first paper, preferences are assumed to be common.

The distribution of the types of individuals (defining their attractiveness or character) is assumed to be discrete.

These distributions are assumed to be independent of sex and the sex ratio is assumed to be one (i.e. the game is symmetric with respect to sex).

Models of Scramble Competition - Continuous Time

Assume that there are n types of prospective partner: $1, 2, \ldots, n$.

There attractiveness of an individual of type i (independent of sex) is denoted v_i, such that $v_1 > v_2 > \ldots > v_n = 1$.

The proportion of individuals that are of type i (independent of sex) is denoted p_i.

The length of the mating season is μ (time measured in terms of potential encounters).

Preferences are common, the payoff of a searcher is assumed to be the attractiveness of the partner obtained (possibly discounted over time), or zero if no partner is found before the end of the mating season.

Models of Scramble Competition - Continuous Time

Individuals find prospective partners according to a non-homogeneous Poisson process, where the rate at which prospective partners are found depends on what proportion of individuals are still searching for a mate.

Let $p(t)$ denote the proportion of the population that are still single at time t.

Let $p_i(t)$ denote the proportion of the population that are still searching at time t and are of type i.

Thus $\sum_{i=1}^{n} p_i(t) = p(t), \quad p_i(0) = p_i$.

Models of Scramble Competition - Continuous Time

Let $\lambda(p)$ [Note $p = p(t)$] denote the rate at which (available) prospective partners are found when the proportion of the population still searching for a mate is p.

Time is scaled so that $\lambda(1) = 1$.

It is assumed that $\lambda(p)$ is non-decreasing in p.

Since p decreasing in time, the rate at which prospective partners are found is non-increasing in time.

Models of Scramble Competition - Continuous Time

Assuming that encounters with members of the opposite sex occur at rate 1.
One might consider two extremes:

a) $\lambda(p) = 1$, i.e. search is concentrated on other singles (the singles bar model).

b) $\lambda(p) = p$, i.e. the probability that an encountered female is single is simply the proportion of females that are single (the randomly mixing population model).

Models of Scramble Competition - Continuous Time

Suppose the population use a given strategy profile such that $A_i(t)$ is the set of mutually acceptable types of prospective partners of a type i player at time t.

Define $\overline{v}_i(t)$ to be the expected value of a prospective partner who is mutually acceptable to a type i searcher and still searching at time t.

Hence,

$$\overline{v}_i(t) = \frac{\sum_{j \in A_i(t)} v_j p_j(t)}{\sum_{j \in A_i(t)} p_j(t)}. \tag{1}$$

Models of Scramble Competition - Continuous Time

Now consider the dynamics of the game under such a strategy profile.

Suppose a player of type i is searching at time t.

The probability that such a player finds a partner in the time interval $[t, t+\delta]$ is given by $\delta\lambda(p)\sum_{j\in A_i(t)} \frac{p_j(t)}{p(t)}$.

Hence,

$$p_i(t+\delta) = p_i(t)\left[1-\delta\lambda(p)\sum_{j\in A_i(t)}\frac{p_j(t)}{p(t)}\right]+O(\delta^2)$$

$$\frac{p_i(t+\delta)-p_i(t)}{\delta} = -p_i(t)\lambda(p)\sum_{j\in A_i(t)}\frac{p_j(t)}{p(t)}+O(\delta).$$

Models of Scramble Competition - Continuous Time

Letting $\delta \to 0$, we obtain the differential equation

$$\frac{dp_i(t)}{dt} = -p_i(t)\lambda(p) \sum_{j \in A_i(t)} \frac{p_j(t)}{p(t)}. \tag{2}$$

This leads to a set of differential equations that define the rate at which prospective partners of each type are found.

Based on this, we can find the best response of an individual of type *i* using dynamic programming (the future expected reward of a searcher is given by an integral expression).

In general, an equilibrium can be estimated using policy iteration.

Models of Scramble Competition - Continuous Time

Suppose the payoff is not discounted over time and there are n levels of attractiveness.

The equilibrium can be described by a sequence of times $(t_1, t_2, t_3, \ldots, t_n)$.

t_i denotes the time at which the most attractive individuals start accepting prospective partners of the i-th level of attractiveness.

$0 = t_1 \leqslant t_2 \leqslant \ldots \leqslant t_{n-1} \leqslant t_n < \mu$, where μ is the length of the breeding season.

Thus the most attractiveness individuals become less choosy as the breeding season progresses.

Models of Scramble Competition - Continuous Time

When $t \leqslant t_i$, individuals of the i-th level of attractiveness are only mutually accepted by other individuals of the i-th level of attractiveness.

For $t \geqslant t_i$, individuals of the i-th level of attractiveness behave in the same way as the most attractive individuals.

Models of Scramble Competition - Continuous Time

When the payoff of a searcher is discounted according to the time at which it finds a partner, the mating patterns are more complicated and multiple equilibria are possible.

Individuals of low attractiveness may become more choosy over some period of time (as in the model of Johnstone).

When $n > 2$, the equilibrium is generally found using an iterative procedure.

Models of Scramble Competition - Continuous Time

When there are two levels of attractiveness and individuals of high attractiveness are rare, then there may be two equilibria.

- 1. Random mating. Each searcher mates with the first prospective partner seen.
- 2. Attractive searchers are initially choosy.

The intuition behind these two equilibira is similar to the intuition behind the existence of multiple equilibria in the game investigated by Burdett and Coles [1999].

Models of Scramble Competition - Continuous Time

When the preferences are homotypic, the form of any equilibrium has been found for any problem where there are two types.

These types can be interpreted as e.g. two sub-species.

Each searcher prefers to mate with an individual of the same sub-species, but gains some reward from pairing with an individual from the other sub-species.

Models of Scramble Competition - Continuous Time

Multiple equilibria can occur in this game when the least common sub-species is more valuable to the more common sub-species as a partner than the most common sub-species is to the less common sub-species.

In this case, there are two equilibria

- 1. Random mating. Each searcher mates with the first prospective partner seen.
- 2. The least common sub-species is initially choosy.

Models of Scramble Competition - Continuous Time

Problems with a larger number of types and homotypic preferences are difficult to solve in general.

For example, suppose there are three types: a common, central type and two rare extreme types on either side of the central type.

It is possible that at time t central types will accept only other central types, whereas the extreme types might be acceptable to each other.

When preferences are common, then at any given time, the set of levels of attractiveness that are mutually acceptable to a searcher of type i are always given by a consecutive sequence of levels $j, j+1, \ldots, i, i+1, \ldots, k-1, k$.

The Problem of Deriving Preferences

Given the pattern of mating, it is very difficult to derive the preferences that led to such a pattern.

This is due to the fact that regardless of whether preferences are common or homotypic, we will observe associative mating, i.e. pairs are formed by individuals who are "similar" to each other.

For example, under common preferences, highly attractive individuals will only pair with other highly attractive individuals.

Individuals of medium attractiveness pair with others of medium attractiveness, not because they prefer such mates, but since they have very little chance of mating with a highly attractive individual.

The Problem of Parental Care

The evolution of mating behaviour is clearly interlinked with the evolution of patterns of parental care.

The amount of parental care supplied by parents of different sexes has a huge influence on the operational sex ratio (OSR).

The OSR is the ratio of the number of males to the number of females in the current mating pool.

This in turn affects strategies of mate choice.

If only females care for offspring, then the OSR is large, females can be choosy and males "compete" with each other.

Parental Care in a Non-Seasonal Breeder

St. Peter's Fish

Parental Care in a Non-Seasonal Breeder -Ramsey (2010)

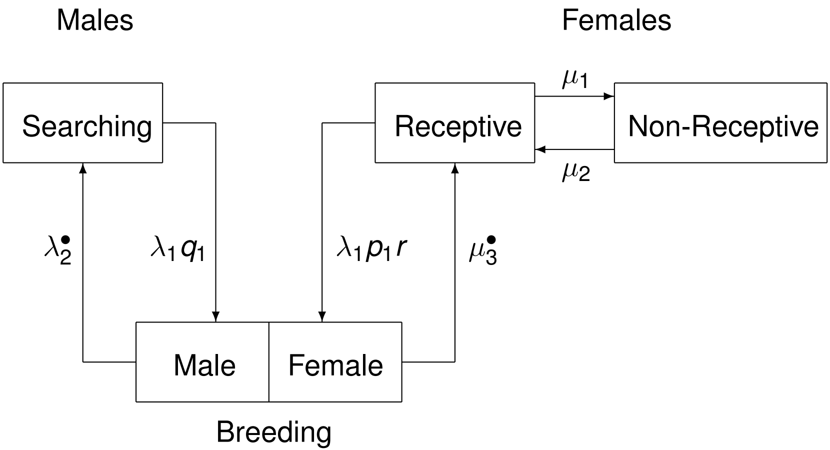

Fig. 1: Transistion rates between states

$\bullet$ represents C or N according to whether a given sex cares for their offspring or not

Parental Care in a Seasonal Breeder

McNamara *et al.* [2000] consider a model of parental care in a seasonally breeding species.

It is assumed that it may be possible to raise several batches of offspring.

After mating, each parent decides whether to care for the offspring or not.

Those who do not care search for another mate.

The rate at which prospective partners are found is an increasing function of the proportion of the opposite sex who are looking for a mate.

Conslusion

Future research on game theoretical models of mate choice should

- 1. Take into account the feedback between mate choice and parental care. In job-search models, this is somewhat analogous to the period of time that people spend in one position.
- 2. Take more into account the possibility of divorce/infidelity. In job-search problems, this means that employed people may be searching for a new job.
- 3. Integrating job-search and mate choice into a life history framework. First individuals invest in obtaining qualifications/growth, before switching to searching for a job or searching for a mate.
- 4. More realistic models of scramble search, e.g. not all individuals start searching at the same time.

Thank You for Your Attention

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