Targeting in social networks

Agnieszka Rusinowska
Centre d'Economie de la Sorbonne, CNRS, Université Paris 1
Paris School of Economics

Outline

Introduction

- Identifying optimal targets is a crucial problem for achieving social impact (marketing, lobbying, voting and political campaigning, etc.).
- Some surveys: Jackson 2008, Acemoglu and Ozdaglar 2011, Bloch 2016, Bramoullé et al. 2016, etc.
- Targeting in networks studied in different fields: complex systems, computer science, mathematics, physics, economics, marketing and organizational science, political science, etc.
- Various methods and techniques: mathematical models, agent-based simulations, some attempts of the empirical (experimental) approach.

Classifications of targeting models

- Non-competitive versus competitive environments: social planner (e.g., monopolist) versus competing persuaders (e.g., oligopoly, perfect competition).
- Perfect information of the network structure versus limited information (e.g., consumers and firms only observe local neighborhods and the degree distribution of the network).
- Our focus: analytical models of targeting in economics, where targets are frequently characterized by new or existing centrality measures.
- Our analysis: Targeting individuals to diffuse information or opinions in a social network; determining and removing a key player to increase/reduce some activities.

Diffusion of information in social networks (1/2)

- How can a firm use social interactions to diffuse information about a new product?
- The firm's problem: selecting a target in the (fixed) social network in order to maximize diffusion.
- Diffusion of information (simple model):
  - Information as a binary signal $(0/1)$ = awareness of a new product, recommendation for the existing product.
  - An agent is in state $0/1$ if he is uninformed/informed or has not/has received a positive recommendation.
  - Agents' states evolve over time according to their interactions with others.
  - The social network g is fixed. At any time, a consumer may receive a message from one of his neighbours.
  - Once informed - he remains informed forever.
  - Diffusion of information is mechanical: a consumer does not control whether he sends information to his neighbors or not.

Diffusion of information in social networks (2/2)

Two main diffusion models:

- The linear threshold model:
  - An agent gets information about the product iff the number of informed neighbours is greater than a certain threshold.
  - If neighbors are heterogeneous, this threshold may be replaced by weighted averages of the status of the agents' neighbors (i.e., some agents are more influential than others, independently of their location in the social network).
- The independent cascade model:
  - Each neighbour of an agent sends information with an independent probability. Agent is informed iff at least one of his neighbours has sent the information.
- Given the (fixed) network structure, targeting = choosing the agent(s) to give first the product in order to diffuse information in the network as fast as possible.

Targeting in computer science and complex systems

- Algorithmic perspective for studying the target selection for the optimal adoption and diffusion of innovation; choosing influential sets of individuals as a problem in discrete optimization.
- Which set of individuals should we target if we aim to have a large cascade of adoptions of a new product or innovation?
- Influence maximization: Domingos and Richardson (2001), Richardson and Domingos (2002), Kempe, Kleinberg and Tardos (2003, 2005).
- Revenue maximization: Hartline, Mirrokni and Sundarajan (2008), Arthur, Motwani, Sharma and Xu (2009).
- Influence maximization with competition: Bharathi, Kempe and Salek (2007), Goyal and Kearns (2012), Dubey, Garg and de Meyer (2006).

Domingos and Richardson (2001)

- First work on targeting algorithms to maximize the probability of sales in a social network.
- Each consumer is of type 0 or 1 (buying the product or not).
- A consumers' probability of buying a product depends on marketing expenditures and the probability that the direct neighbours have bought the product.
- How can a firm optimally target consumers by directing marketing expenditures to specific agents?
- Three algorithms:
  - a single-pass algorithm (one iteration),
  - a greedy algorithm (which increases marketing expenditures when they increase payoffs),
  - hill-climbing algorithm (increases expenditures where matters most).

Kempe, Kleinberg and Tardos (2003)

- The firm selects an initial set A of k nodes in the social network in order to maximize the total number of informed nodes.
- Information is diffused according to a linear threshold or independent cascade model.
- Results:
  - The optimal targeting problem is NP-hard (in general impossible to find a polynomial algorithm to compute the optimal target set A except in special cases, e.g., in Richardson and Domingos (2002)'s linear model where the optimal target is a solution of a system of linear equations).
  - Determining the approximation bound on the efficiency of the hill-climbing algorithm, which selects agents to place in the set A by looking sequentially at agents with the highest influence.

Targeting and pricing in economics and operations research

- Many studies of targeting in the economics literature characterize targets by existing and new centrality measures.
- Ballester et al. 2006; Banerjee et al. 2013, 2017; Candogan et al. 2012, Tsakas 2014, 2017, Bimpikis et al. 2016, Galeotti et al. 2017, Demange 2017, Grabisch et al. 2018, Rusinowska and Taalaibekova 2019, ...
- Consumption externalities:
  - Monopoly pricing with consumption externalities: Candogan et al. 2012, Bloch and Querou 2013, Fainmesser and Galeotti 2013, ...
  - Competitive pricing in social networks: Banerji and Dutta 2009, Galeotti 2010, Jullien 2011, ...
- We will briefly discuss some of these works.

Outline

Matrix representation of a network

- A network is represented by a graph (N, g), where
  - $N = \{1, 2, ..., n\}$ set of nodes (agents, players, vertices)
  - $g = [g_{ij}]$ real-valued $n \times n$ matrix (adjacency matrix)
- g_{ij} - relationship between i and j (possibly weighted and/or directed), also referred to as a link ij or an edge
- We assume that graphs are simple, i.e., $g_{ii} = 0$ for all $i \in N$ (no loops).
- In what follows we consider an unweighted and undirected network:

$$g_{ij} = \begin{cases} 1 & \text{if there is a link between } i \text{ and } j \\ 0 & \text{otherwise,} \end{cases}$$

and $g_{ij} = g_{ji}$ for all $i, j \in N$.

Walks and paths

- How can one node be reached from another one in g?
  - Walk = sequence of links $i_1 i_2, \cdots, i_{K-1} i_K$ such that $g_{i_k i_{k+1}} = 1$ for each $k \in \{1, \cdots, K-1\}$
    (a node or a link may appear more than once)
  - Trail = walk in which all links are distinct
  - Path = trail in which all nodes are distinct
- Geodesic between two nodes is a shortest path between them.
- $d(i,j;g)$ = geodesic distance between i and j in g
  If there is a path between i and j in g, then
  $d(i,j;g)$ = the number of links in a shortest path between i and j

$$d(i,j;g) = \min_{\text{paths } P \text{ from } i \text{ to } j} \sum_{(k,l)\in P} g_{kl}.$$

If there is no path between i and j in g, we set $d(i,j;g) = \infty$.

Measuring centrality and prestige

- Given nodes that represent agents (players) and links that represent relationships between the agents (communication, influence, dominance ...), the following questions may appear:
  - How central is a node (player) in the network?
  - What is his position and prestige?
  - How influential is his opinion?
  - To which degree is the agent successful and powerful in collective decision making?
  - ...
- Centrality measures can be useful for the analysis of the information flows, bargaining power, infection transmission, influence, etc.
- Different centrality measures capture different aspects of centrality, and therefore can have highest values for different individuals.

Standard measures of centrality

- The concept of centrality captures a kind of prominence of a node in a network.
- Since the late 1940's a variety of different centrality measures that focus on specific characteristics inherent in prominence of an agent have been developed.
- Measures of centrality can be categorized into the following main groups (Jackson (2008)):
  1. Degree centrality - how connected a node is
  2. Closeness centrality - how easily a node can reach other nodes
  3. Betweenness centrality - how important a node is in terms of connecting other nodes
  4. Prestige- and eigenvector-related centrality - how important, central, or influential a node's neighbors are.

Degree and closeness centrality measures

- The degree centrality (Shaw (1954), Nieminen (1974)): How connected is a node in terms of direct connections?
  - The degree centrality $C_i^d(g)$ of node i in network g is given by

$$C_i^d(g) = \frac{d_i(g)}{n-1} = \frac{|N_i(g)|}{n-1} \in [0,1]$$

  - Index of the node's communication activity: the more ability to communicate directly with others, the higher the centrality.
- The closeness centrality (Beauchamp (1965), Sabidussi (1966)) is based on proximity: How easily can a node reach other nodes?
  - The closeness centrality $C_i^c(g)$ of node i in network g is

$$C_i^c(g) = \frac{n-1}{\sum_{j \neq i} d(i,j;g)}$$

  - Measure of the node's independence or efficiency: the possibility to communicate with others depends on a min. nb of intermediaries.

Betweenness centrality

- The betweenness centrality (Bavelas (1948), Freeman (1977, 1979)): How important is a node in terms of connecting other nodes?
- The betweenness centrality $C_i^b(g)$ of node i in network g is

$$C_i^b(g) = \frac{2}{(n-1)(n-2)} \sum_{k \neq j : i \notin \{k,j\}} \frac{P_i(kj)}{P(kj)}$$

$P_i(kj)$ = number of geodesics between k and j containing $i \notin \{k, j\}$
$P(kj)$ = total number of geodesics between k and j

- Index of the potential of a node for control of communication: the possibility to intermediate in the communications of others is of importance.

Katz prestige

- Measures of centrality that are based on the idea that a node's importance is determined by the importance of its neighbours.
- The Katz prestige $C_i^{PK}(g)$ of node i in g is defined as

$$C_i^{PK}(g) = \sum_{j \neq i} g_{ij} \frac{C_j^{PK}(g)}{d_j(g)}$$

If j has more relationships, then i gets less prestige from being connected to j. This definition is self-referential.

- Calculating $C^{PK}(g)$ - finding the unit eigenvector of $\widetilde{g}$:

$$C^{PK}(g) = \widetilde{g}\, C^{PK}(g), \quad (\mathbb{I} - \widetilde{g}) C^{PK}(g) = \mathbf{0}$$

$\widetilde{g}$ - the normalized adjacency matrix g, $\widetilde{g}_{ij} = \frac{g_{ij}}{d_j(g)}$, $\widetilde{g}_{ij} = 0$ for $d_j(g) = 0$. $C^{PK}(g)$ - the $n \times 1$ vector of $C_i^{PK}(g)$, $i \in N$, $\mathbb{I}$ - the $n \times n$ identity matrix, $\mathbf{0}$ - the $n \times 1$ vector of 0's.

Eigenvector centrality

- If we do not normalize g, we get the eigenvector centrality $C^e(g)$ associated with g (Bonacich (1972)).
- The centrality of a node is proportional to the sum of the centrality of its neighbours.

$$\lambda C_i^e(g) = \sum_j g_{ij} C_j^e(g)$$

$$\lambda C^e(g) = g C^e(g)$$

  and thus $C^e(g)$ is an eigenvector of g and λ is the corresponding largest eigenvalue of matrix g.
- The Katz prestige can be seen as a kind of eigenvector centrality with the network adjacency matrix being weighted.

Second prestige measure of Katz

- $C^{PK2}(g,a)$ = the second prestige measure of Katz (1953)
- Introducing an attenuation parameter a to adjust the measure for the lower 'effectiveness' of longer walks in a network.
- The prestige of a node is a weighted sum of the walks that emanate from it, and a walk of length k is of worth a^k, where $0 < a < 1$. The vector of prestige of nodes is

$$C^{PK2}(g,a) = ag\mathbf{1} + a^2g^2\mathbf{1} + \cdots + a^kg^k\mathbf{1} + \cdots$$

  where $\mathbf{1}$ is the $n \times 1$ vector of 1's.
- Each entry of the vector $g^k\mathbf{1}$ is the total number of walks of length k that emanate from each node; $g\mathbf{1}$ is the vector of degrees of nodes.
- For a sufficiently small, $C^{PK2}(g,a)$ is finite and

$$C^{PK2}(g,a) - agC^{PK2}(g,a) = ag\mathbf{1}, \quad C^{PK2}(g,a) = (\mathbb{I} - ag)^{-1}ag\mathbf{1}.$$

Bonacich centrality

- A two-parameter family of prestige measures which can be seen as a direct extension of $C^{PK2}(g,a)$.
- An agent can have some status which does not depend on its connections to others.
- Bonacich centrality (Bonacich (1987)) is given by

$$C^B(g,a,b) = ag\mathbf{1} + abg^2\mathbf{1} + \cdots + ab^k g^{k+1}\mathbf{1} + \cdots$$

$$C^B(g,a,b) = (\mathbb{I} - bg)^{-1}\, ag\mathbf{1}$$

  where a and b are parameters, and b is sufficiently small.
  - b captures how the value of being connected decays with distance.
  - a captures the base value on each node.
  - For $b = 0$, $C^B(g,a,b)$ takes into account only walks of length 1 and reduces to $ad_i(g)$.
  - For $b > 0$, $C^B(g,a,b)$ takes into account more distant interactions.
  - $C^{PK2}(g,a)$ and $C^B(g,a,b)$ coincide when $a = b$.

Outline

The point of departure and objectives

- Grabisch, Mandel, Rusinowska and Tanimura (2018):
  - extension of DeGroot (1974) by introducing two (stubborn) external players with extreme opinions who compete over the prominence in the network of non-strategic players
  - non-cooperative game played by the external players, the impact of the strategic aspects on the characterization of the key target
- Rusinowska and Taalaibekova (2019):
  - introducing the centrist persuader with a specific position that represents balance, neutrality, and equal combination of the extreme positions (e.g., three-candidate political or university elections)
  - studying the impact of the centrist persuader on the competition between the two extremist persuaders (opinion convergence and consensus reaching, characteristics of the targets)

The model with three persuaders (1/2)

- $N = \{1, \dots, n\}$ society of individuals discussing a certain issue
- Each individual i has an initial opinion represented by $x_i(0) \in [0, 1]$ and interpreted as the intensity of i's opinion "yes" at time 0
- With no intervention, the individuals update their opinion as in DeGroot (1974), i.e., there is a $n \times n$ row-stochastic matrix of weights $W = [w_{ij}]$, where w_{ij} is the weight (trust) that i places on the opinion of j and:

$$\mathbf{x}(t) = W\mathbf{x}(t-1) = W^t\mathbf{x}(0)$$

  where $\mathbf{x}(t) = (x_1(t), \dots, x_n(t))'$ is the opinion vector at time t
- W irreducible: $\forall\ i, j \in N\ \exists\ m(i,j)$ such that $w_{ij}^{(m(i,j))} > 0$

The model with three persuaders (2/2)

- The society is observed by three persuaders A, B and C with the fixed opinions 1, $\frac{1}{2}$ and 0, respectively.
- Each persuader chooses one individual in N (s_A, s_B and s_C) to form a link with in order to influence the opinion formation in the society.
- A, B and C are also characterized by possibly unequal (positive) persuasion impacts λ, γ and μ to adjust influence in the society.
- The same adjustment of influence holds for s_B and s_C being targeted by B and C, with impacts γ and μ, respectively.

Walks, cycles and weights of walks

- We associate to W a directed graph Γ on N such that there is an arc (i, j) from i to j (meaning that i listens to j) iff $w_{ij} > 0$.
- A walk from i to k is a sequence of nodes $(i_1 = i, i_2, \cdots, i_{j-1}, i_j = k)$ s.t. $w_{i_m i_{m+1}} > 0$ for each $m \in \{1, \cdots, j-1\}$.
- A cycle around i is a walk from i to i which does not pass through i between the starting and ending nodes.
- For any walk $p = (i_1, \ldots, i_j)$:
  $w(p)$ = "weight" of p measured according to W

$$w(p) := \prod_{m=1}^{j-1} w_{i_m, i_{m+1}}$$

- C_i^j = set of cycles around i that pass through j, B_i^j = set of walks that start from any node $\neq i$, end up in i and go through j

The influenceability and intermediacy

$$\text{For } i, j \in N,\ \ i \neq j, \quad c_i^j := \sum_{p \in \mathcal{C}_i^j} w(p), \quad b_i^j := \sum_{p \in \mathcal{B}_i^j} w(p)$$

- c_i^j = sum of weights of cycles around i that pass through j = probability for i to be reached by the influence of j before he receives the self-feedback (echo) of his own opinion
- $d_i c_i^j$ = *influenceability of individual* i, given that j is targeted by another persuader, with d_i being i's out-degree
- b_i^j = *intermediacy (influence, centrality) of* j *relatively to* i = sum of weights of walks to i that pass through j = sum of the probabilities for all agents other than i to be reached by the influence of j before this of i

The extended matrix of influence

One gets a new $(n+3)\times(n+3)$ influence matrix

$$M_{\lambda,\gamma,\mu}(\mathbf{s}) = \left[\begin{array}{ccc|c} 1 & 0 & 0 & \mathbf{0} \\ 0 & 1 & 0 & \mathbf{0} \\ 0 & 0 & 1 & \mathbf{0} \\ \hline \multicolumn{3}{c|}{\Delta_{\lambda,\gamma,\mu}(\mathbf{s})E_{\lambda,\gamma,\mu}(\mathbf{s})} & \Delta_{\lambda,\gamma,\mu}(\mathbf{s})W \end{array}\right]$$

where

- the weight renormalization matrix $\Delta_{\lambda,\gamma,\mu}(\mathbf{s})$ is a diagonal matrix with elements $\frac{d_1}{d_1+\lambda\delta_{1,s_A}+\gamma\delta_{1,s_B}+\mu\delta_{1,s_C}}, \cdots, \frac{d_n}{d_n+\lambda\delta_{n,s_A}+\gamma\delta_{n,s_B}+\mu\delta_{n,s_C}}$
- the strategic influence matrix is $E_{\lambda,\gamma,\mu}(\mathbf{s}) = \left[\frac{\lambda}{d_{s_A}}e_{s_A} \; \frac{\gamma}{d_{s_B}}e_{s_B} \; \frac{\mu}{d_{s_C}}e_{s_C}\right]$
- e_i is the unit vector with coordinate 1 at i, δ is the Kronecker symbol: $\delta_{i,s_j} = 1$ if $i = s_j$ for $i \in N$, $s_j \in \{s_A, s_B, s_C\}$ and 0 otherwise.

The updating rule under the persuaders' intervention

- The vector of opinions is extended to $\mathbf{x}(t) = [1 \; \frac{1}{2} \; 0 \; \mathbf{x}_N(t)]'$.
- The opinion updating rule is now determined by

$$\mathbf{x}(t+1) = M_{\lambda,\gamma,\mu}(\mathbf{s})\mathbf{x}(t) = (M_{\lambda,\gamma,\mu}(\mathbf{s}))^{t+1}\mathbf{x}(0)$$

which leads to the evolution law for the opinions of the individuals in N given by

$$\mathbf{x}_N(t+1) = \Delta_{\lambda,\gamma,\mu}(\mathbf{s})E_{\lambda,\gamma,\mu}(\mathbf{s}) \begin{bmatrix} 1 \\ \frac{1}{2} \\ 0 \end{bmatrix} + \Delta_{\lambda,\gamma,\mu}(\mathbf{s})W\mathbf{x}_N(t)$$

Convergence of opinions

When the society gets a new persuader, the one with the centrist position, the opinion convergence is preserved in the society.

Proposition 1

For any initial vector of opinions $\mathbf{x}(0) := [1\ \frac{1}{2}\ 0\ \mathbf{x}_N(0)]'$, we have

$$\lim_{t \to +\infty} (M_{\lambda,\gamma,\mu}(\mathbf{s}))^t \begin{bmatrix} 1 & \frac{1}{2} & 0 & \mathbf{x}_N(0) \end{bmatrix}' = \begin{bmatrix} 1 & \frac{1}{2} & 0 & \overline{\mathbf{x}}_N(\mathbf{s}) \end{bmatrix}'$$

where

$$\overline{\mathbf{x}}_N(\mathbf{s}) = [I - \Delta_{\lambda,\gamma,\mu}(\mathbf{s})W]^{-1} \Delta_{\lambda,\gamma,\mu}(\mathbf{s}) \left(\frac{\lambda}{d_{s_A}} e_{s_A} + \frac{\gamma}{2 d_{s_B}} e_{s_B} \right)$$

Consensus reaching (1/3)

If the three persuaders choose the same target, then the long run opinion in the society converges towards a consensus $\alpha \in [0,1]$ which is determined by the three persuasion impacts.

Proposition 2

If $s_A = s_B = s_C$, then the individuals in N reach a consensus α given by

$$\alpha = \frac{2\lambda + \gamma}{2(\lambda + \gamma + \mu)}$$

In particular, if $\lambda = \mu$, then the consensus is $\alpha = \frac{1}{2}$.

Extension to a multi-target framework: This result holds independently of the number of the same targets, i.e., when the three persuaders can choose several targets for diffusion of information.

Consensus reaching (2/3)

$$\alpha = \frac{2\lambda + \gamma}{2(\lambda + \gamma + \mu)}$$

- When $\gamma \to 0$, the consensus is $\frac{\lambda}{\lambda+\mu}$ (Grabisch et al., 2018).
- When $\gamma \to +\infty$ and $\lambda, \mu \in \mathbb{R}_+$, the consensus tends to $\frac{1}{2}$.
- When $\lambda \to +\infty$ and $\gamma, \mu \in \mathbb{R}_+$, the consensus tends to 1.
- When $\mu \to +\infty$ and $\lambda, \gamma \in \mathbb{R}_+$, the consensus tends to 0.
- $\frac{2\lambda + \gamma}{2(\lambda + \gamma + \mu)} > \frac{\lambda}{\lambda + \mu}$ if and only if $\lambda < \mu$
  Under the same target, the presence of B always improves the situation of the weaker extreme persuader (moves the consensus closer to the opinion of the persuader with the smaller impact).

Consensus reaching (3/3)

When persuaders A and C target the same individual, then the society ends up in a consensus, even if the centrist persuader targets another individual and independently of his own impact, but only if the extreme persuaders are equally strong. In this case, the consensus is equal to $\frac{1}{2}$.

Proposition 3

If $s_A = s_C$ and $\lambda = \mu$ then the individuals in N reach a consensus $\alpha = \frac{1}{2}$.

Game played by the persuaders (1/3)

- We consider a game $\mathcal{G}_{\lambda,\gamma,\mu}$ played between the three persuaders, with their set of strategies being N.
- Aim of each persuader: bringing the asymptotic average opinion as close as possible to the own opinion (1 for A, $\frac{1}{2}$ for B, 0 for C)
- The game-theoretic model is a system of minimization problems: given a strategy profile $\mathbf{s}=(s_A, s_B, s_C) \in N \times N \times N$

$$\pi^A_{\lambda,\gamma,\mu}(s_A, s_B, s_C) = \left(1 - \frac{1}{n}\mathbf{1}'\overline{\mathbf{x}}_N(\mathbf{s})\right)^2$$

$$\pi^B_{\lambda,\gamma,\mu}(s_A, s_B, s_C) = \left(\frac{1}{2} - \frac{1}{n}\mathbf{1}'\overline{\mathbf{x}}_N(\mathbf{s})\right)^2$$

$$\pi^C_{\lambda,\gamma,\mu}(s_A, s_B, s_C) = \left(\frac{1}{n}\mathbf{1}'\overline{\mathbf{x}}_N(\mathbf{s})\right)^2$$

Game played by the persuaders (2/3)

- We denote the aggregate opinion formed in the society by

$$\tilde{x}_N(\mathbf{s}) := \mathbf{1}'\overline{\mathbf{x}}_N(\mathbf{s}) = \sum_{i \in N} \overline{x}_i(\mathbf{s})$$

where $\overline{\mathbf{x}}_N(\mathbf{s}) = [I - \Delta_{\lambda,\gamma,\mu}(\mathbf{s})W]^{-1}\Delta_{\lambda,\gamma,\mu}(\mathbf{s})\left(\frac{\lambda}{d_{s_A}}e_{s_A} + \frac{\gamma}{2d_{s_B}}e_{s_B}\right)$

Theorem 1

The payoffs of persuaders A, B and C, given the strategy profile (s_A, s_B, s_C) are as follows:

(i) If $s_A = s_B = s_C = i$, then

$$\tilde{x}_N(i, i, i) = \frac{n(2\lambda + \gamma)}{2(\lambda + \gamma + \mu)}$$

Game played by the persuaders (3/3)

Theorem 1 (continued)

(ii) If $s_A = s_C = i$ and $s_B = k \neq i$, then:

$$\tilde{x}_N(i,k,i) = \frac{2\lambda(\gamma b_k^i + d_k c_k^i n) + \gamma((\lambda+\mu) b_i^k + d_i c_i^k n)}{2(\gamma d_i c_i^k + (\lambda+\mu)(d_k c_k^i + \gamma))}$$

(iii) If $s_A = k$ and $s_B = s_C = i \neq k$, then:

$$\tilde{x}_N(k,i,i) = \frac{2\lambda((\gamma+\mu) b_i^k + d_i c_i^k n)) + \gamma(\lambda b_k^i + d_k c_k^i n)}{2(\lambda d_i c_i^k + (\gamma+\mu)(d_k c_k^i + \lambda))}$$

(iv) If $s_A = s_B = i$ and $s_C = k \neq i$, then:

$$\tilde{x}_N(i,i,k) = \frac{(2\lambda+\gamma)(\mu b_k^i + d_k c_k^i n)}{2(\mu d_i c_i^k + (\lambda+\gamma)(d_k c_k^i + \mu))}$$

Equal persuasion impacts (1/2)

- Let $\lambda = \gamma = \mu$. We use the simplified notations $\mathcal{G}_\lambda$, π^A_λ, π^B_λ, and π^C_λ.

Theorem 2

A profile of strategies (i, i, i) is an equilibrium of the game $\mathcal{G}_\lambda$ if and only if for all $k \in N \setminus \{i\}$

$$b^i_k - 2b^k_i \geq \frac{n}{\lambda}\left(d_i c^k_i - d_k c^i_k\right)$$

- In the extended three-persuader model, the condition to reach the equilibrium (i, i, i) requires more from the intermediacy of i over k than under the two extreme persuaders: i must be even more influential (central) among others to compensate the impact of the two other persuaders.

Equal persuasion impacts (2/2)

- As the number of individuals in the society increases, the relative importance of intermediacy compared to influenceability goes down.
- Conversely, the relative importance of intermediacy goes up with the level of λ, the impact of the persuaders.

Proposition

(i) (i, i, i) is an equilibrium of $\mathcal{G}_\lambda$ as $\lambda \to 0$ if and only if for all $k \in N$

$$d_k c_k^i \geq d_i c_i^k$$

(ii) (i, i, i) is an equilibrium of $\mathcal{G}_\lambda$ as $\lambda \to +\infty$ if and only if for all $k \in N$

$$b_k^i \geq 2 b_i^k$$

Unequal persuasion impacts: If $\gamma, \mu > 0$ are fixed and the impact λ of A is sufficiently large, then $\mathcal{G}_{\lambda,\gamma,\mu}$ has only equilibria in mixed strategies.

Examples (1/2)

- Perfectly symmetric society: $d_i = d_k$, $c_i^k = c_k^i$, $b_i^k = b_k^i$.
  While (i, i, i) was always an equilibrium in the model with two persuaders, the equilibrium condition does not hold here, but the game has non-symmetric NE. E.g., for $n = 3$ there exist 12 NE: 6 profiles (i, j, i) with $i \neq j$ leading to the consensus $\frac{1}{2}$, 6 profiles (i, j, k) with $i \neq j \neq k$ and $i \neq k$ for which $\tilde{x}_N(i, j, k) = \frac{3}{2}$ but the individual opinions are different from each other.
- Star society: $d_i = n - 1$, $d_k = 1$ for all $k \neq i$.
  $c_k^i = 1$, $c_i^k = \frac{1}{n-1}$, $b_k^i = n - 1$, $b_i^k = 1$.
  The equilibrium condition for (i, i, i) is always satisfied (unless $n < 3$). E.g., NE for $n = 3$ are $(2, 2, 2)$ and 16 non-symmetric NE: 6 profiles (i, j, i) with $i \neq j$, 6 profiles (i, j, k) with $i \neq j \neq k$ and $i \neq k$, and 4 other NE in which two "neighbouring" persuaders target the center, i.e., $(1, 2, 2)$, $(2, 2, 1)$, $(2, 2, 3)$ and $(3, 2, 2)$.
  (i, i, i) lead to the consensus $\frac{1}{[illegible]}$, while (i, j, k) are s.t. $\tilde{x}_N(i, j, k) = \frac{3}{[illegible]}$

Examples (2/2)

- Directed circle: $d_i = 1$ for every $i \in N$, for any $k \neq i$, $c_i^k = c_k^i = 1$, $b_k^i = l(k,i)$, $b_i^k = l(i,k)$, where $l(k,i)$, $l(i,k)$ are the lengths of the (unique) shortest walk from k to i, and from i to k, respectively. If $\lambda = \gamma = \mu$ then no pure strategy symmetric NE exists, as in the case with two persuaders. Similarly, non-symmetric equilibria do not exist.
- Line network: Two types of nodes: $d_1 = d_n = 1$ and $d_j = 2$ for each $j \neq 1, n$. No symmetric equilibrium (i,i,i) exists, but the game admits non-symmetric NE, e.g., for $n = 4$: $(2,3,2)$, $(3,2,3)$, $(2,4,2)$ and $(3,1,3)$, all leading to the consensus $\frac{1}{2}$. Under equilibrium the extreme persuaders target one of the "middle" individuals while the centrist persuader chooses either another "middle" individual or the "end" individual which is not the neighbour of the extreme persuaders' target.

Conclusion (1/2)

- Consensus can emerge if the three persuaders target the same individual. Additionally under the equal impacts, the centrist one has no effect on the social opinion, but the outcome is ideal for him.
- The existence of a pure strategy NE depends on the network structure (e.g., no NE in circular networks).
- A symmetric equilibrium in pure strategies emerges when the persuaders exert an equal impact. It is characterized by two features of the targets: influenceability and intermediacy (centrality).
- With three persuaders, the relative influence of a potential target must be at least twice higher than the one of any other individual in the network (the persuaders are demanding higher centrality from the target to compensate the impact of the additional persuader).

Conclusion (2/2)

- Influenceability gains importance versus intermediacy as the size of the network grows or the impact of the persuaders decreases.
- When the persuasion impacts are unequal, the high-impact persuader aims at ensuring preeminence on the network by increasing his centrality and diminishing the influenceability of his opponents' target.
- The low-impact persuaders seek to keep a minimal level of influence by hiding their target from the opponent's impact.
- A growing number of the persuaders does not affect too much the game when the persuaders are weak.

Outline

Consumption externalities

- Network externalities = agents' consumption of a good is affected by the number of agents consuming the same good (Katz and Shapiro 1985, Farrell and Saloner 1985).
- Network externalities arise in many environments, e.g., in telecommunications (agents benefit from others using the same communication device), in the software industry (development of application is driven by the number of users), etc.
- Early works model the consumption externality as global phenomenon rather than network based feature (the valuation of consumers is defined as a function of the total number of users).
- More recent models study consumption externalities based on a given social network.

Candogan et al. 2012, Bloch and Querou 2013 (1/5)

Candogan, Bimpikis and Ozdaglar (*Operations Research* 2012)
Bloch and Querou (*Games and Economic Behavior* 2013)

- The society consists of a set of n agents embedded in a social network represented by the adjacency matrix $G = [g_{ij}]$, where g_{ij} measures the influence of agent j on i's consumption, $g_{ij} \geq 0$, $g_{ii} = 0$ for all i.
- A monopolist introduces a divisible good in the market and chooses a vector $\mathbf{p}$ of prices, $(p_1, \ldots, p_n)$, from the set of pricing strategies $\mathbf{P}$, where p_i is the price offered by the monopolist to agent i for one unit of the good.

Candogan et al. 2012, Bloch and Querou 2013 (2/5)

- The utility of agent i is given by the following quadratic expression:

$$u_i(q_1, \dots, q_n, p_i) = a_i q_i - \frac{1}{2} b_i q_i^2 + q_i \sum_j g_{ij} q_j - p_i q_i$$

  where $q_i \geq 0$ is the amount of the good that agent i consumes.
- u_i increases with the consumption q_j of direct neighbors j.
- The positive externality implies that, given any vector of prices $\mathbf{p}$, the consumption levels are computed as the equilibrium of a non-cooperative games played among consumers.

Candogan et al. 2012, Bloch and Querou 2013 (3/5)

- Two-stage pricing-consumption game:
  - Stage 1 (Pricing): The monopolist chooses the pricing strategy $(p_1, \ldots, p_n)$ in order maximize profits:

    $$\Pi = \sum_i (p_i - c) q_i$$

    where c denotes the marginal cost of producing a unit of the good, and q_i denotes the amount of the good that i purchases in the second stage of the game.
  - Stage 2 (Consumption): Agent i chooses to purchase q_i units of the good as to maximize his utility given the prices chosen by the monopolist and $\mathbf{q}_{-i}$.
- We look for the subgame-perfect equilibria of the two-stage pricing-consumption game.

Candogan et al. 2012, Bloch and Querou 2013 (4/5)

- Two assumptions:
  (A1) $2b_i > \sum_j g_{ij}$ for all i (the optimal consumption level of each agent is bounded)
  (A2) $a_i > c$ for all i (when the monopolist sets prices optimally, all consumers purchase a positive amount of the good)
- Under (A1) and (A2), the optimal price vector is given by:

$$\mathbf{p} = \mathbf{a} - (\mathbf{\Lambda} - \mathbf{G}) \left(\mathbf{\Lambda} - \frac{\mathbf{G} + \mathbf{G}^T}{2} \right)^{-1} \frac{\mathbf{a} - c\mathbf{1}}{2}$$

where Λ is a diagonal matrix with terms b_i on the diagonal, and $\mathbf{a}$ is the vector of a_i's.

Candogan et al. 2012, Bloch and Querou 2013 (5/5)

- Corollary: Under (A1) and (A2), when the matrix G is symmetric (influence is undirected), the monopoly sets a uniform price:

$$\mathbf{p} = \frac{\mathbf{a} + c}{2}$$

  i.e., the optimal prices do not depend on the network structure.
- The authors obtain an alternative characterization of the optimal prices (using the Bonacich centrality).
- The optimal price consists of three components:
  - (i) a nominal term that is independent of the network structure,
  - (ii) a discount term proportional to the influence that the agent exerts over the network (quantified by the agent's Bonacich centrality),
  - (iii) a markup term proportional to the influence that the network exerts on the agent.

Ballester, Calvo-Armengol and Zenou (2006) (1/5)

Ballester, Calvo-Armengol and Zenou (*Econometrica 2006*)

- They study a model in which individuals located in a network choose actions (criminal activities) which affect the payoffs of other individuals within the network.
- Which individuals should be eliminated from the network if the objective is to minimize crime?
- A noncooperative (network) game with local payoff complementarities.
- Each player $i = 1, \ldots n$ selects an effort $x_i \geq 0$ and gets the utility:
  $u_i(x_1, \cdots, x_n) = \alpha_i x_i + \frac{1}{2}\sigma_{ii} x_i^2 + \sum_{j \neq i} \sigma_{ij} x_i x_j$
  which is strictly concave in own effort ($\frac{\partial^2 u_i}{\partial x_i^2} = \sigma_{ii} < 0$).

Ballester, Calvo-Armengol and Zenou (2006) (2/5)

$$u_i(x_1, \cdots, x_n) = \alpha_i x_i + \frac{1}{2}\sigma_{ii}x_i^2 + \sum_{j \neq i} \sigma_{ij} x_i x_j$$

- We set $\alpha_i = \alpha$ and $\sigma_{ii} = \sigma < 0$ for all i.
- Bilateral influences are captured by the cross-derivatives: $\frac{\partial^2 u_i}{\partial x_i \partial x_j} = \sigma_{ij}$ for $i \neq j$.
- If $\sigma_{ij} > 0$, then efforts of i and j are strategic complements from i's perspectives (an increase in j's effort triggers a upwards shift in i's response).
- If $\sigma_{ij} < 0$ then efforts of i and j are strategic substitutes from i's perspectives.

Ballester, Calvo-Armengol and Zenou (2006) (3/5)

- Let $\sum = [\sigma_{ij}]$ be the square matrix of cross-effects, also used for denoting the simultaneous move n-player game with the strategy space $\mathbb{R}_+$. It is additively decomposed into:

$$\sum = -\beta \mathbf{I} - \gamma \mathbf{U} + \lambda \mathbf{G}$$

  where $\mathbf{I}$ is the n-square identity matrix, $\mathbf{U}$ is the n-square matrix of ones, $\mathbf{G} = [g_{ij}]$ is a zero-diagonal nonnegative adjacency matrix of the network g of relative payoff complementarities across pairs, $\beta > 0$, $\gamma \geq 0$, $\lambda > 0$.
- Bilateral influences result from the combination of an idiosyncratic effect, a global interaction, and a local interaction.

Ballester, Calvo-Armengol and Zenou (2006) (4/5)

- Consider a network g with adjacency n-square matrix $\mathbf{G} = [g_{ij}]$ and a scalar $a \geq 0$ such that $[m_{ij}(g,a)] = [\mathbf{I} - a\mathbf{G}]^{-1}$ is well defined and non-negative, where $m_{ij}(g,a) = \sum_{k=0}^{\infty} a^k g_{ij}^{[k]}$ and

$$b_i(g,a) = \sum_{j=1}^{n} m_{ij}(g,a)$$

  measures total number of walks starting in i.
- $\mathbf{b}(g,a)$ is obtained from the Bonacich centrality by an affine transformation:

$$\mathbf{b}(g,a) = \mathbf{1} + C^{PK2}(g,a)$$

  where $C^{PK2}(g,a)$ is the second prestige of Katz.

Ballester, Calvo-Armengol and Zenou (2006) (5/5)

- For the matrix $[\beta\mathbf{I} - \lambda\mathbf{G}]^{-1}$ well defined and nonnegative, the game $\sum$ has a unique Nash equilibrium (NE).
- The NE action of a player is proportional to his Bonacich centrality.
- In order to decrease aggregate effort, the key player (to target) is identified by an intercentrality measure which takes into account i's centrality and his contribution to the centrality of the others:

$$c_i(g, a) = \frac{b_i(g, a)^2}{m_{ii}(g, a)}$$

- While the Bonacich centrality of i counts the number of walks in g that stem from i, the intercentrality counts the total number of such walks that hit i: it is the sum of i's Bonacich centrality and i's contribution to every other player's Bonacich centrality.

Galeotti and Goyal (2009) (1/3)

Galeotti and Goyal (*The RAND Journal of Economics* 2009)

- They model networks in terms of degree distributions and study influence strategies in the presence of local interaction.
- They consider two groups of players, where the first group M (the firm) chooses a strategy with the aim of influencing members of the second group N to choose certain actions.
- Optimal influence strategies will target individuals with low or high connectivity, depending on the nature of the interaction.

Galeotti and Goyal (2009) (2/3)

- The firm only knows the degree distribution of consumers in the network and consumers' degrees.
- A targeted marketing strategy assigns a different advertising expenditure on each consumer on the basis of his degree.
- Each consumer: out-/indegree (number of influencing/influenced agents).
- The profit of reaching a consumer with (out)degree k and spending marketing expenditures x is represented by $\phi_k(x)$.
- $\phi_k(x)$ exhibits increasing (decreasing) marginal returns in degree if, for any $x > x'$, $\phi_{k+1}(x) - \phi_{k+1}(x') > (<) \phi_k(x) - \phi_k(x')$.

Galeotti and Goyal (2009) (3/3)

- Two models:
  - one step diffusion version of the independent cascade model with $\phi_k(x) = 1 - (1 - x)^{k+1}$
  - one step version of the threshold model with $\phi_k(x) = (1 - x)xk^{1-\beta}$ with $\beta > 1$.
- Monotonicity of the optimal targeting policy: if $\phi_k(x)$ exhibits increasing/decreasing marginal returns to degree (threshold model/ cascade model), nodes with higher/lower degree receive more advertising.

Banerjee, Chandrasekhar, Duflo and Jackson (2013, 2019) (1/2)

Banerjee, Chandrasekhar, Duflo and Jackson (*Science* 2013, *Review of Economic Studies* 2019)

- Identifying the most influential agents in a gossip process.
- Players generate some information about particular people, which is then stochastically passed from neighbour to neighbour.
- Diffusion centrality of a node i in a network with an adjacency matrix g, a probability p of passing the information and T iterations is defined as the ith entry of the vector

$$DC(g,p,T) = \Big[\sum_{t=1}^{T}(pg)^t\Big]\mathbf{1}$$

which measures how extensively the information spreads from i.

Banerjee, Chandrasekhar, Duflo and Jackson (2013, 2019) (2/2)

- Consider T iterations of information passing from a single initially informed node i where at each iteration every informed node tells each neighbor with probability p. The diffusion centrality of node i then corresponds to the expected total number of times that all nodes taken together get the information.
- If $T = 1$, diffusion centrality is proportional to degree centrality.
- As $T \to +\infty$, it becomes proportional to either Katz-Bonacich centrality or eigenvector centrality, depending on whether p is smaller or greater than the inverse of the first eigenvalue of the adjacency matrix g.
- In the intermediate region of T, diffusion centrality differs from existing measures.

Galeotti, Golub and Goyal (2017)

- Individuals interact strategically with their network neighbours.
- A planner's goal: maximizing welfare or minimizing volatility.
- The planner can shape incentives to achieve his goal.
- A method of decomposing any potential intervention into principal components determined by the network.
- A particular ordering of principal components describes the planner's priorities across a range of network intervention problems.
- If actions are strategic complements, the optimal intervention changes all agents' incentives in the same direction in proportion to their eigenvector centralities.
- If actions are strategic substitutes, the optimal intervention moves neighbours' incentives in opposite directions.

Bimpikis, Ozdaglar and Yildiz (2016) (1/2)

Bimpikis, Ozdaglar and Yildiz (*Operations Research* 2016)

- A game-theoretic model where two strategic competing firms seek to optimally allocate their marketing budgets to maximize the product awareness resulting from social influence.
- Agents receive a message that can either be one of the two brands or the status quo.
- The probabilities of the different messages are determined by the agents' awareness levels for each of the two brands.
- The authors provide a characterization of the optimal targeted advertising strategies and highlight their dependence on the underlying social network structure.

Bimpikis, Ozdaglar and Yildiz (2016) (2/2)

- They provide conditions under which it is optimal for the firms to asymmetrically target a subset of the individuals.
- At equilibrium firms invest inefficiently high in targeted advertising and the extent of the inefficiency is increasing in the centralities of the agents they target.
- Agent i's absorption centrality = expected number of visits to node i before absorption at either node $n+1$, $n+2$, or $n+3$ (corresponding to the two firms and the status quo) for a random walk started at a node other than i.
- The limiting average awareness levels are a weighted sum of the firms' marketing efforts where the weights are given by the absorption centralities of the agents.

References (1/9)

Acemoglu D, Ozdaglar A (2011) Opinion dynamics and learning in social networks, *Dynamic Games and Applications* 1:3–49

Arthur D, Motwani R, Sharma A, Xu Y (2009) Pricing strategies for viral marketing on social networks, mimeo, Stanford University

Ballester C, Calvo-Armengol A, Zenou Y (2006) Who's who in networks. Wanted: The key player, *Econometrica* 74(5):1403–1417

Banerjee A, Chandrasekhar A, Duflo E, Jackson M (2013) Diffusion of *Science* 341, 1236498

Banerjee A, Chandrasekhar A, Duflo E, Jackson M (2019) Using gossips to spread information: Theory and evidence from a randomized controlled trial, Forthcoming in *Review of Economic Studies*

Banerji A, Dutta B (2009) Local network externalities and market segmentation, *International Journal of Industrial Organization* 27:605–614

References (2/9)

Bavelas B (1948) A mathematical model for group structure, *Human Organizations* 7:16–30

Beauchamp MA (1965) An improved index of centrality, *Behavioral Science* 10:161–163

Bharathi S, Kempe D, Salek M (2007) Competitive influence maximization in social networks, WINE

Bimpikis K, Ozdaglar A, Yildiz E (2016) Competitive targeted advertising over networks, *Operations Research* 64(3):705–720

Bloch F (2016) Targeting and pricing in social networks, In: Bramoullé et al. (2016)

Bloch F, Querou N (2013) Pricing in social networks, *Games and Economic Behavior* 80:263–281

References (3/9)

Bonacich PB (1972) Factoring and weighting approaches to status scores and clique identification, *Journal of Mathematical Sociology* 2:113–120

Bonacich PB (1987) Power and centrality: a family of measures, *American Journal of Sociology* 92:1170–1182

Bramoullé Y, Galeotti A, Rogers B (2016) *The Oxford Handbook of the Economics of Networks*, Oxford University Press

Campbell A (2013) Word of mouth and percolation in social networks, *American Economic Review* 103:2466–2498

Candogan 0, Bimpikis K, Ozdaglar A (2012) Optimal pricing in networks with externalities, *Operations Research* 60(4):883–905

Demange G (2017) Optimal targeting strategies in a network under complementarities, *Games and Economic Behavior* 105:84–103

References (4/9)

Dodds P, Watts D (2007) Influentials, networks, and public opinion formation, *Journal of Consumer Research* 34(4):441–458

Domingos P, Richardson M (2001) Mining the network value of customers, *Proceedings of the 7th Conference on Knowledge Discovery and Data Mining*, 57–66

Dubey P, Garg R, de Meyer B (2014) Competing for customers in a social network, *Journal of Dynamics and Games* 1(3):377–409

Fainmesser I, Galeotti A (2014) The value of network information, Mimeo, Brown University and University of Essex

Farrell J, Saloner G (1985) Standardization, compatibility and innovation, *RAND Journal of Economics* 16:70–83

Freeman LC (1977) A set of measures of centrality based on betweenness, *Sociometry* 40:35–41

References (5/9)

Freeman LC (1979) Centrality in social networks: Conceptual clarification, *Social Networks* 1:215–239

Galeotti A (2010) Talking, searching and pricing, *International Economic Review* 51:1159–1174

Galeotti A, Golub B, Goyal S (2017) Targeting interventions in networks, Cambridge-INET Working Paper Series 2017/21, Cambridge Working Papers in Economics 1744

Galeotti A, Goyal S (2009) Influencing the influencers: A theory of strategic diffusion, *RAND Journal of Economics* 40:509–532

Goldenberg J, Han S, Lehmann D, Hong JW (2009) The role of hubs in the adoption process, *Journal of Marketing* 73:1–13

Goldenberg J, Libai B, Muller E (2001) Talk of the network: A complex systems look at the underlying process of word-of-mouth, *Marketing Letters* 12:211–223

References (6/9)

Goyal S (2007) *Connections: An Introduction to the Economics of Networks*, Princeton University Press

Goyal S, Heidari H, Kearns M (2019) Competitive contagion in networks, *Games and Economic Behavior* 113:58–79

Goyal S, Kearns M (2012) Competitive contagion in networks, *STOC 2012*, long version in *Games and Economic Behavior*

Grabisch M, Mandel A, Rusinowska A, Tanimura E (2018) Strategic influence in social networks, *Mathematics of Operations Research* 43(1):29–50

Hartline J, Mirrokni V, Sundarajan M (2008) Optimal marketing strategies over social networks, *Proceedings of WWW*, Beijing, China, 189–198

Jackson MO (2008) *Social and Economic Networks*, Princeton University Press

References (7/9)

Jullien B (2011) Competing in multi-sided markets: Divide and conquer, *American Economic Journal: Microeconomics* 3:186–219

Katz L (1953) A new status index derived from sociometric analysis, *Psychometrika* 18:39–43

Katz M, Shapiro C (1985) Network externalities, competition and compatibility, *American Economic Review* 75:424–440

Kempe D, Kleinberg J, Tardos E (2003) Maximizing the spread of infuence through a social network, *Proceedings of the 9th International Conference on Knowledge Discovery and Data Mining*, 137–146

Kempe D, Kleinberg J, Tardos E (2005) Influential nodes in a diffusion model for social networks, *Proceedings of 32nd International Colloquium on Automata, Languages and Programming*, 1127–1138

References (8/9)

Libai B, Muller E, Peres R (2013) Decomposing the value of word-of-mouth seeding programs: Acceleration versus expansion, *Journal of Marketing Research* 50:161–176

Nieminen J (1974) On centrality in a graph, *Scandinavian Journal of Psychology* 15:322–336

Richardson R, Domingos P (2002) Mining knowledge-sharing sites for viral marketing, *KDDM* 02

Rusinowska A, Taalaibekova A (2019) Opinion formation and targeting when persuaders have extreme and centrist opinions, *Journal of Mathematical Economics* 84:9–27

Sabidussi G (1966) The centrality index of a graph, *Psychometrika* 31:581–603

Shaw ME (1954) Group structure and the behaviour of individuals in small groups, *Journal of Psychology* 38:139–149

References (9/9)

Stephen A, Dover Y, Goldenberg J (2010) A comparison of the effects of transmitter activity and connectvity on the diffusion of information in social networks, Mimeo, INSEAD

Stonedahl F, Rand W, Wilensky U (2010) Evolving viral marketing strategies, *GECCO 10*

Tsakas N (2017a) Optimal influence under observational learning, Working Paper

Tsakas N (2017b) Diffusion by imitation: The importance of targeting agents, *Journal of Economic Behavior and Organization* 139:118–151

Wasserman S, Faust K (1994) *Social Network Analysis: Methods and Applications*, Cambridge University Press, Cambridge

Watts D (2002) A simple model of global cascades in random networks, *Proccedings of the National Academy of Sciences* 99:5766–5771