The Shapley value in voting and cooperative contexts: a hindsight and some extensions

Josep Freixas

Dep. de Matemàtiques de la Universitat Politècnica de Catalunya

Kempten Autumn Talks 2020
November, 30.

The main actors of this talk

The contexts:

- Cooperative games
- Simple games
- Voting games with several ordered levels of approval
- Multichoice games

The main actors:

Cooperative games

Definition (Cooperative game)

The pair (N, v) is a cooperative game (in characteristic function form) if N is a finite player set and $v : 2^N \to \mathbb{R}$ is a characteristic function that assigns to every coalition $S \subseteq N$ an attainable payoff $v(S)$ such that $v(\emptyset) = 0$.

Comments:

- Each coalition of players is a payoff or "worth".
- Equilibrium concepts are developed to describe how these values are allocated to the members of these coalitions.
- Only the attainable payoff values are described.
- Payoffs are transferable in the sense that payments can be made to the individual members.

Two properties:

- Monotonicity: if for all $S, T \in 2^N$

$$S \subset T \text{ implies } v(S) \leq v(T)$$

- Superadditivity: if for all $S, T \in 2^N$ with $S \cap T = \emptyset$:

$$v(S) + v(T) \leq v(S \cup T)$$

Special types of cooperative games:

- Symmetric game: if the value $v(S)$ only depends on the number of players $|S|$ in the coalition S. Hence, there is some $f : N \to \mathbb{R}$ such that $v(S) = f(|S|)$ for all $S \subseteq N$
- Simple game: if for each coalition $S \subseteq N$ we have either $v(S) = 0$ or $v(S) = 1$.

The class of all cooperative games is a linear vector space:

For all games $v, \ w \in \mathcal{G}_N$ and scalars $\lambda, \ \mu \in \mathbb{R}$ the linear combination $\lambda v + \mu w$ is defined as

$$(\lambda v + \mu w)(S) = \lambda v(S) + \mu w(S)$$

verifies $\lambda v + \mu w \in \mathcal{G}_N$.

Dimension:

Dimension$(\mathcal{G}_N) = 2^n - 1$. $(\mathcal{G}_N \approx \mathbb{R}^{2^n-1})$.

Two basis:

- The standard basis of a Euclidean space, $b_S \in \mathcal{G}_N$.
  $\mathcal{B} = \{b_S : S \subseteq N, S \neq \emptyset\}$.

$$b_S(T) = \begin{cases} 1 & \text{if } T = S \\ 0 & \text{if } T \neq S \end{cases}$$

Now, $v = \sum_{S \neq \emptyset} v(S) b_S$.

- The unanimity basis of a Euclidean space, $u_S \in \mathcal{G}_N$.
  $\mathcal{B} = \{u_S : S \subseteq N, S \neq \emptyset\}$.

$$u_S(T) = \begin{cases} 1 & \text{if } S \subseteq T \\ 0 & \text{otherwise} \end{cases}$$

Now, $v = \sum_{S \neq \emptyset} \Delta_v(S) u_S$, where $\Delta_v(S) = \sum_{T \subseteq S} (-1)^{s-t} v(T)$

The Shapley value (cooperative games)

Shapley (1953) established the value theory: a single-valued solution concept that is guaranteed to exist for every game and which is completely characterized by a given set of behavioral axioms.

Definition

A value is a function $\phi : \mathcal{G}_N \to \mathbb{R}^n$ that assigns to every cooperative game a single allocation $\phi(v) \in \mathbb{R}^n$.

Game theorists developed axiomatic values on certain specific classes of cooperative games $\mathcal{H} \subseteq \mathcal{G}_N$. If restrictions on a domain of a value are applied, one usually indicates these solutions as indices rather than values. Today I will mention the Shapley-Shubik index (Shapley and Shubik, 1954).

A test for a new value

The seminal axiomatic value introduced by Shapley (1953) has had a very **special appeal**.

1. + Shapley characterized this value through very appealing axioms,
2. + Shapley provided a compelling probabilistic model,
3. + This value has a wide ranging applicability,
4. + The value's computation is rather straightforward and can be done in multiple ways.

Shapley value axioms:

A value $\psi : \mathcal{G}_N \to \mathbb{R}^n$:

- is efficient if for every game $v \in \mathcal{G}_N$

$$\sum_{i \in N} \psi_i(v) = v(N)$$

- satisfies the null-player property for every $v \in \mathcal{G}_N$ it holds $\psi_i(v) = 0$ for every null-player $i \in N$ in the game v.
- is symmetric if for every permutation $\rho : N \to N$

$$\psi_{\rho(i)}(\rho v) = \psi_i(v)$$

- is additive if for all games $v, w \in \mathcal{G}_N$ and every player $i \in N$:

$$\psi_i(v + w) = \psi_i(v) + \psi_i(w)$$

Appealing axiomatic characterization

Theorem (Shapley, 1953)

The Shapley value ϕ is the unique value on $\mathcal{G}_N$ that satisfies:

- *efficiency,*
- *the null-player property,*
- *symmetry, and*
- *additivity.*

Some formulations:

- Standard formulation:

$$\phi_i(v) = \sum_{S \in 2^{N \setminus \{i\}}} \frac{s!(n-s-1)!}{n!}(v(S \cup \{i\}) - v(S))$$

- The probabilistic formulation:

$$\phi_i(v) = \frac{1}{n!} \sum_{\pi \in \Pi}((v(pr_\pi(i)) \cup \{i\}) - v(pr_\pi(i)))$$

- The Harsanyi dividends formulation:

$$\phi_i(v) = \sum_{S \subseteq N \,:\, i \in S} \frac{\Delta_v(S)}{|S|}$$

- The multilinear formulation:

$$\phi_i(v) = \int_0^1 f_i(t, t, \ldots, t)\, dt, \quad \text{and others...}$$

Example (N, v): A three player cooperative game.

S	$\emptyset$	1	2	3	12	13	23	123
$v(S)$	0	6	0	0	10	10	10	21

Table: The game defined by its characteristic function.

Computation of the Shapley value by using marginal contributions.

$$\begin{aligned}
\phi_1(v) &= \tfrac{1}{3}v(1) + \tfrac{1}{6}(v(12) + v(13)) + \tfrac{1}{3}(v(123) - v(23)) \\
&= \tfrac{1}{3}(6) + \tfrac{1}{6}(10 + 10) + \tfrac{1}{3}(21 - 10) = 9 \\
\phi_2(v) &= \tfrac{1}{6}(v(12) + v(23) - v(1)) + \tfrac{1}{3}(v(123) - v(13)) \\
&= \tfrac{1}{6}(4 + 10) + \tfrac{1}{3}(21 - 10) = 6 \\
\phi_3(v) &= \tfrac{1}{3}(v(13) + v(23) - v(1)) + \tfrac{1}{6}(v(123) - v(12)) \\
&= \tfrac{1}{6}(4 + 10) + \tfrac{1}{3}(21 - 10) = 6
\end{aligned}$$

Example (N, v): A three player cooperative game.

S	$\emptyset$	1	2	3	12	13	23	123
$v(S)$	0	6	0	0	10	10	10	21

Table: The game defined by its characteristic function.

Computation of the Shapley value as an expected value of marginal contributions.

permutation	1	2	3
123	6	4	11
132	6	11	4
213	10	0	11
231	11	0	10
312	10	11	0
321	11	10	0
sums	54	36	36

$$\phi(v) = \frac{1}{6}(54, 36, 36) = (9, 6, 6)$$

Example (N, v): A three player cooperative game.

S	$\emptyset$	1	2	3	12	13	23	123
$v(S)$	0	6	0	0	10	10	10	21

Table: The game defined by its characteristic function.

Computation of the Shapley value by the Harsanyi dividends formulation.

S	$\emptyset$	1	2	3	12	13	23	123
$v(S)$	0	6	0	0	10	10	10	21
$\Delta_v(S)$	0	6	0	0	4	4	10	-3

$$\phi_1(v) = \frac{6}{1} + \frac{4+4}{2} + \frac{-3}{3} = 9$$

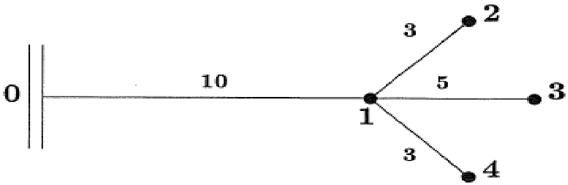
$$\phi_2(v) = \frac{0}{1} + \frac{4+10}{2} + \frac{-3}{3} = 6$$

$$\phi_3(v) = \frac{0}{1} + \frac{4+10}{2} + \frac{-3}{3} = 6$$

Applications of the Shapley value

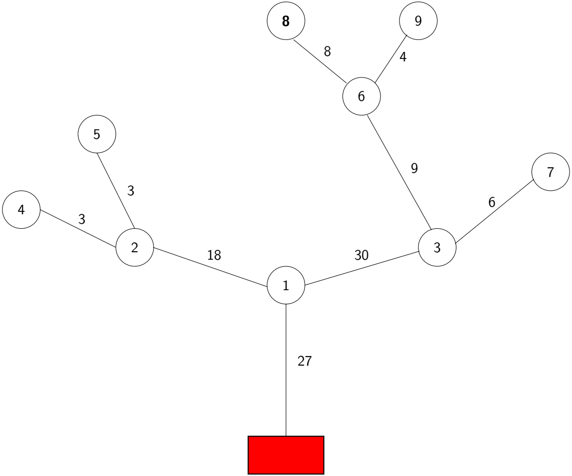
- Bankruptcy games
- Saving games
- Cost games
- Games defined by (graphs) networks
- Clan games
- Games connecting to a source using a tree
- Warehouse game
- Airport game
- Real state market game

An application: Electrical Network



$$c(1) = 10,\ c(2) = 13,\ c(3) = 15,\ c(4) = 13,\ c(12) = 13,$$
$$c(13) = 15,\ c(14) = 13,\ c(23) = 18,\ c(24) = 16,\ c(34) = 18,$$
$$c(123) = 18,\ c(124) = 16,\ c(134) = 18,\ c(234) = 21,\ c(1234) = 21$$

$$\phi(c) = (2.5,\ 2.5 + 3,\ 2.5 + 5,\ 2.5 + 3) = (2.5,\ 5.5,\ 7.5,\ 5.5)$$



How to distribute the expenses among the 9 neighbors?

Payments:

	1	2	3	4	5	6	7	8	9
They all use the first edge	3	3	3	3	3	3	3	3	3
Only 2, 4 and 5 use the edge with a value of 18		6		6	6				
Only 4 uses the edge with a value of 3				3					
Only 5 uses the edge with a value of 3					3				
Only 3, 6, 7, 8 i 9 use the edge with a value of 30			6			6	6	6	6
Only 7 uses the edge with a value of 6							6		
Only 6, 8 i 9 use the edge with a value of 9						3		3	3
Only 8 uses the edge with a value of								8	
Only 9 uses the edge with a value of 4									4
Payments: $\phi(c)$	3	9	9	12	12	12	15	**20**	16

Simple games

Definition (Simple game seen as a cooperative game)

The pair (N, v) is a simple game if N is a finite set of players and $v : 2^N \to \{0, 1\}$ is a characteristic function that assigns to every coalition $S \subseteq N$ an attainable payoff $v(S)$ such that $v(\emptyset) = 0$.

Definition (Simple game seen as a voting game)

The pair (N, v) is a simple game if N is a finite set of voters and $v : 2^N \to \{\mathit{lose}, \mathit{win}\}$ is a function that classifies coalitions into winning and losing coalitions, $\emptyset$ is declared to be losing and N to be winning.

- Monotonicity is usually required in both definitions.

Most authors do not distinguish between the two situations, and the most of the examples they use belong to the political context.

The Shapley-Shubik index (simple games)

Shapley and Shubik (1954) defined their power index on simple games as the restriction of the Shapley value to simple games.

Comments:

- Is it the same to divide revenues or costs among players than evaluate the influence that voters have in making-decisions?
- They did not provide an axiomatization for the index.
  Dubey (1975) established the first axiomatization of the index by replacing additivity by transferability.

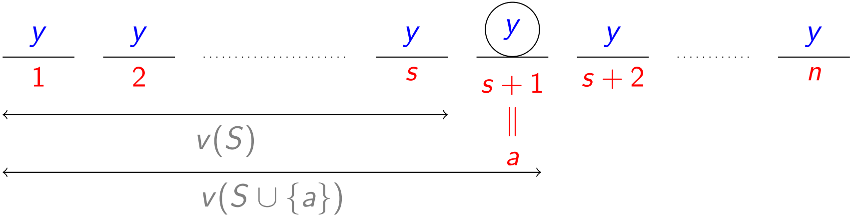
The probabilistic model for the Shapley value (cooperative games)

1. Assume that all orderings of players are equally probable.
2. Assume that, in his/her turn, everybody agrees to form the grand coalition.
3. The player receives the marginal contribution to the coalition of his/her predecesors in arriving to the queue.

- The Shapley value is the expected value, under uniform probability, of the marginal contributions in the space of all orderings (permutations)

A probabilistic model for the Shapley value (cooperative games)

Marginal contribution of player *a* in a given permutation:

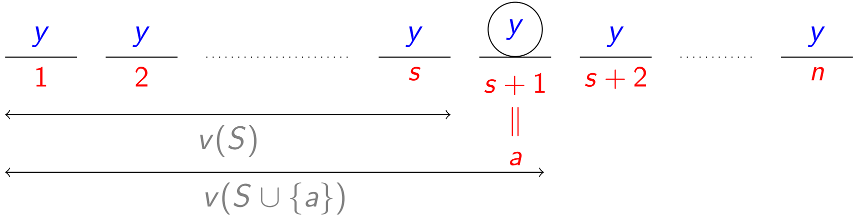


- <u>Comment</u>: Rationality of players + good properties stimulate the formation of N. Shapley proposes to divide the revenues or costs according to his value.

- <u>Question</u>: Is the probabilistic model provided a convincing argument for cooperative games?

A probabilistic model for the Shapley value (cooperative games)

Marginal contribution of player a in a given permutation:

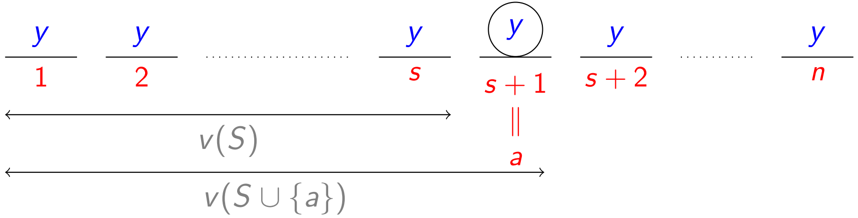


- Comment: Rationality of players $+$ good properties stimulate the formation of N. Shapley proposes to divide the revenues or costs according to his value.

- Question: Is the probabilistic model provided a convincing argument for cooperative games? yes

A probabilistic model for the Shapley value (cooperative games)

Marginal contribution of player a in a given permutation:

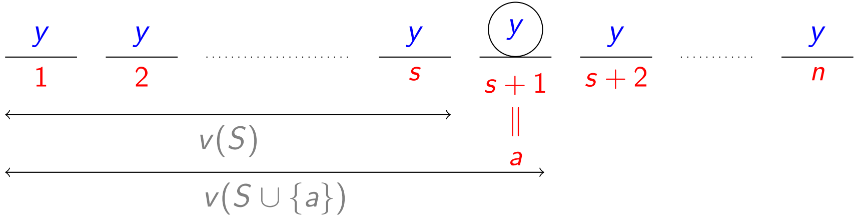


- Comment: Rationality of players + good properties stimulate the formation of N. Shapley proposes to divide the revenues or costs according to his value.

- Question: Is the probabilistic model provided a convincing argument for cooperative games? yes
  and for the "restriction" to simple games?

A probabilistic model for the Shapley value (cooperative games)

Marginal contribution of player *a* in a given permutation:



- Comment: Rationality of players + good properties stimulate the formation of N. Shapley proposes to divide the revenues or costs according to his value.

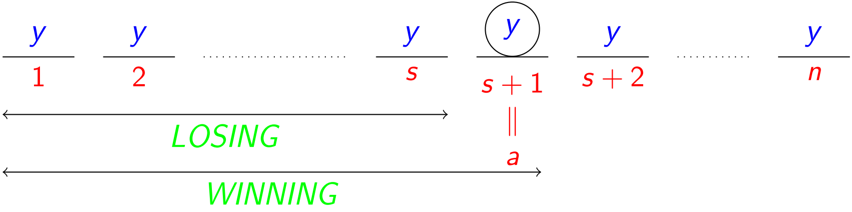
- Question: Is the probabilistic model provided a convincing argument for cooperative games? yes
  and for the "restriction" to simple games? ...maybe "not"

The probabilistic model for the S&S power index, 1954

1. Assume that all orderings of voters are equally probable.
2. Assume that everybody votes "yes" in his/her turn.
3. A player is "pivotal" if the coalition of his/her predecessors in the permutation is losing and his/her addition to it does the new coalition winning.

- The S&S index is the probability of being pivotal under de above scheme, or equivalently,
- it is the expected value of the marginal contributions, under uniform probability, in the set of all permutations.

The probabilistic model for the S&S power index, 1954

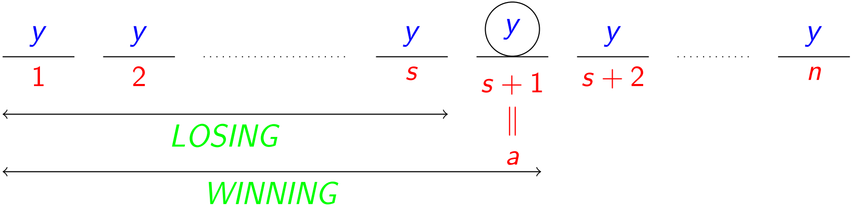


Voter a is the pivot for the given permutation because

$$\{1, 2, \ldots, s\} \text{ loses} \quad \text{and} \quad \{1, 2, \ldots, s, s+1\} \text{ wins}$$

Is this model satisfactory?

The probabilistic model for the S&S power index, 1954



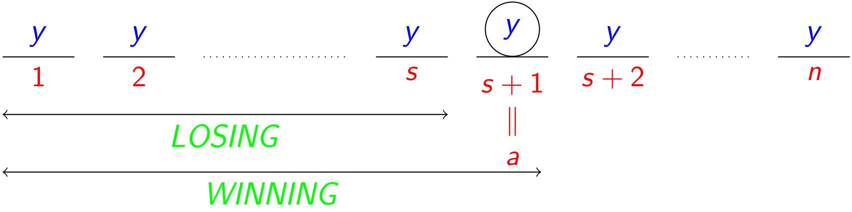
Voter a is the pivot for the given permutation because

$$\{1, 2, \ldots, s\} \text{ loses} \quad \text{and} \quad \{1, 2, \ldots, s, s+1\} \text{ wins}$$

Is this model satisfactory?

- It is, if the simple game is a TU cooperative game.

The probabilistic model for the S&S power index, 1954



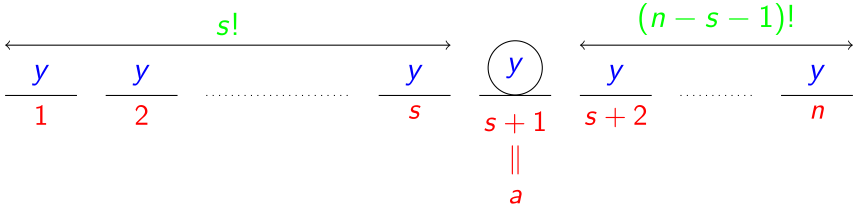
Voter a is the pivot for the given permutation because

$$\{1, 2, \ldots, s\} \text{ loses} \quad \text{and} \quad \{1, 2, \ldots, s, s+1\} \text{ wins}$$

Is this model satisfactory?

- It is, if the simple game is a TU cooperative game.
- Possibly not, if the simple game modelizes a voting system. Why?

Standard formula for both S-value and SS-index
(cooperative and simple games)



Well-known formula by taking common factors:

The Shapley value ϕ is given by

$$\phi_a(v) = \sum_{S \in 2^{N \setminus \{a\}}} \frac{s!(n-s-1)!}{n!}[v(S \cup \{a\}) - v(S)]$$

for any $a \in N$, where $s = |S|$.

Probabilistic model by Shapley

In summary,

- Shapley provides a probabilistic convincing model for his value, and
- exports it to simple games, even if the simple game modelizes a voting system.

Cooperative games $\longrightarrow$ Simple games

A test for a new power index

The seminal axiomatic power index introduced by Shapley and Shubik (1954) has had a very special appeal.

1. - + Dubey (1975) characterized this index through appealing axioms. Einy and Haimanko (2011) use appealing axioms without using efficiency.
2. - Shapley and Shubik did not provide a compelling probabilistic model when measuring power as influence.
3. + - This index has a wide ranging applicability, although the Banzhaf index is perhaps best appropriated index for measuring power as influence.
4. + The index's computation is rather straightforward and can be done in multiple ways.

An alternative probabilistic model for simple and cooperative games

The new ideas I intend in this talk:

- Show a more “convincing” probabilistic model for the *S&S* index for simple games, seen as voting systems.
- Export the probabilistic model to cooperative games.
- Extend the index to wider models than simple games and cooperative games.

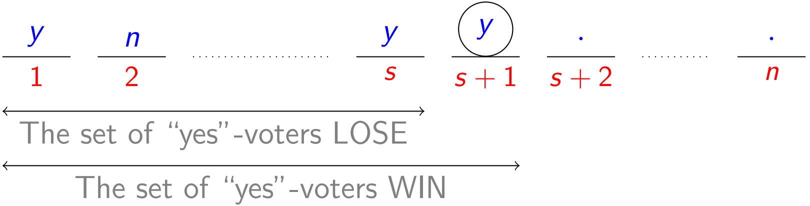
Simple games $\longrightarrow$ Cooperative games

An alternative probabilistic model for simple games

We now consider roll-calls or queues ($n!2^n$) instead of permutations ($n!$):

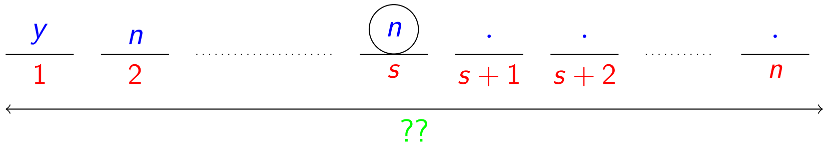
In her turn (in the permutation) the voter, additionally, decides either to vote "in favor" or "against" the proposal submitted at hand.

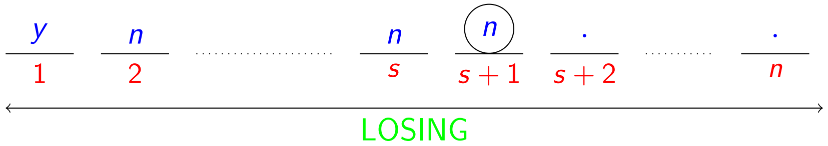
Pivotal voter by voting "yes" (simple games)



An alternative probabilistic model for simple games

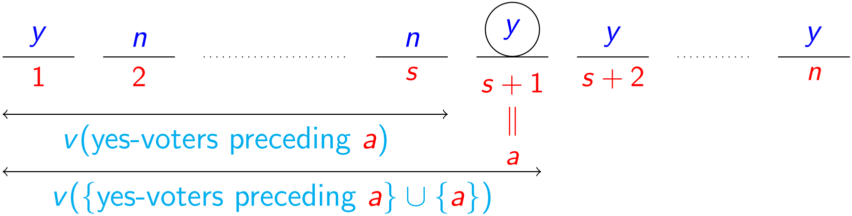
Pivotal voter by voting "no":





No matter how the subsequent voters are going to vote.

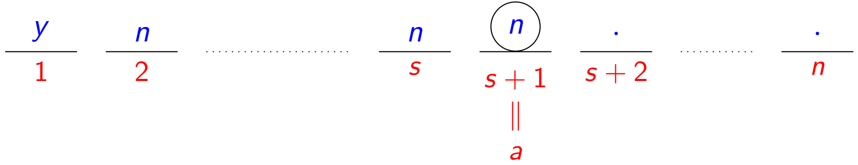
Marginal contribution when deciding forming part of the coalition:



Marginal contribution of player a:

$$v(\{\text{yes-voters preceding } a\} \cup \{a\}) - v(\{\text{yes-voters preceding } a\})$$

Marginal contribution threatening not be part of the coalition:



Marginal contribution of player a:

$$v(\{\text{yes-players preceding } a\} \cup \{\text{all players after } a\} \cup \{a\}) -$$

$$v(\{\text{yes-players preceding } a\} \cup \{\text{all players after } a\})$$

Example:

$n = 5$ **players, the number of queues is** $5! \cdot 2^5 =$ **3840**:

Consider the queue given by the permutation 12345 and choices *nynyy*:

$$\frac{n}{1} \quad \frac{y}{2} \quad \frac{n}{3} \quad \frac{y}{4} \quad \frac{y}{5}$$

Players' marginal contributions:

$$(v(12345) - v(2345), v(2), v(2345) - v(245), v(24) - v(2), v(245) - v(24))$$

Felsenthal and Machover alternative bargaining model, 1996:

Felsenthal and Machover (1996) proved that $\phi_a(v)$ is the expected value of $M(v, R, a)$ in the probability space of roll-calls, $\mathcal{R}$. For any voter $a \in N$:

$$\phi_a(v) = \frac{1}{2^n \cdot n!} \sum_{R \in \mathcal{R}} M(v, R, a).$$

Their proof was done by checking that the above "value" satisfies the three axioms by Shapley and quoted that:

> *The direct proof(*) implies a combinatorial fact that is certainly non-trivial, and may be of some independent interest.*

(*) The coincidence between this formula and the one by Shapley.

...and they also wrote:

If one attempts to prove previous formula by showing directly that the standard Shapley formula for simple games coincides with the proposed above formula (previous slide), one encounters rather formidable combinatorial difficulties. This suggests that our Theorem and Corollary are a disguised form of a combinatorial fact that is certainly non-trivial.

Felsenthal and Machover
Public Choice (1996)

New formula for the S-value and S&S-index directly derived from the alternative probabilistic model

Theorem (Bernardi and Freixas, 2018)

The Shapley value ϕ *is given by*

$$\phi_a(v) = \sum_{S \in 2^{N \setminus \{a\}}} \Gamma^n(s)[v(S \cup \{a\}) - v(S)]$$

for any $a \in N$ *, where* $s = |S|$ *and* $s = 0, \ldots, n-1$

$$\Gamma^n(s) = \frac{1}{2^n n!}\left[s! \sum_{k=0}^{s} \frac{(n-k-1)!}{(s-k)!} 2^k + (n-s-1)! \sum_{k=0}^{n-s-1} \frac{(n-k-1)!}{(n-s-1-k)!} 2^k \right]$$

Goal of the proof: $\Gamma^n(s) = \dfrac{s! \cdot (n-s-1)!}{n!}$

Two techniques for the proof: Double induction. Generating functions.

Does the new formula mean a real progress?

- – It does not. In simple or cooperative games the given formula is equivalent to the traditional one. But...

Does the new formula mean a real progress?

- − It does not. In simple or cooperative games the given formula is equivalent to the traditional one. But...
- + It solves the Felshental and Machover posed problem in Public Choice, 1996.

Does the new formula mean a real progress?

- − It does not. In simple or cooperative games the given formula is equivalent to the traditional one. But...
- \+ It solves the Felshental and Machover posed problem in Public Choice, 1996.
- \+ It complements the "alternative" probabilistic model which is clearly more convincing for simple games, seen as voting games.

Does the new formula mean a real progress?

- − It does not. In simple or cooperative games the given formula is equivalent to the traditional one. But...
- + It solves the Felshental and Machover posed problem in Public Choice, 1996.
- + It complements the "alternative" probabilistic model which is clearly more convincing for simple games, seen as voting games.
- + Nevertheless, an extension of the Shapley-Shubik index for $(j, 2)$ simple games (Freixas and Zwicker, 2003) is obtained by extending the second alternative approach and <u>I do not see the form</u> to obtain it by only using the traditional probabilistic model.

Does the new formula mean a real progress?

- − It does not. In simple or cooperative games the given formula is equivalent to the traditional one. But...
- + It solves the Felshental and Machover posed problem in Public Choice, 1996.
- + It complements the "alternative" probabilistic model which is clearly more convincing for simple games, seen as voting games.
- + Nevertheless, an extension of the Shapley-Shubik index for $(j, 2)$ simple games (Freixas and Zwicker, 2003) is obtained by extending the second alternative approach and I do not see the form to obtain it by only using the traditional probabilistic model.
- + Moreover, the "new" formula derived is also useful for computing an extension of the "Shapley value" for the larger class of j-cooperative games or multichoice cooperative games (Hsiao and Raghavan, 1992).

We have seen:

Simple games $\longrightarrow$ Cooperative games

We have seen:

Simple games $\longrightarrow$ Cooperative games

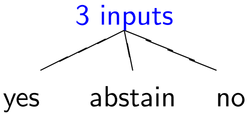
Our purpose is now twofold:

Simple games $\longrightarrow$ $(j, 2)$-simple games

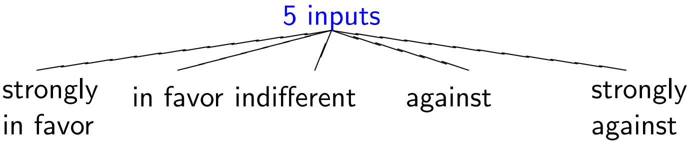
Cooperative games $\longrightarrow$ Multichoice games

Games with several ordered levels of approval

A $(3,2)$**-voting rule:** (or $(3,2)$-simple game)



A $(5,2)$**-voting rule:** (or $(5,2)$-simple game)



$(j, 2)$-voting rules $(j > 2)$

$j = 3$: Tripartitions:

$$(S_1, S_2, S_3) \in 3^N$$

- S_1 set of "yes" voters
- S_2 set of "abstainers"
- S_3 set of "no" voters

Monotonicity:

$$(S_1, S_2, S_3) \sqsubseteq (T_1, T_2, T_3)$$

if

$$S_1 \subseteq T_1 \quad \text{and} \quad S_2 \subseteq T_1 \cup T_2$$

A $(3,2)$ simple game (N, V) consists of a finite set N of voters together with a value function $V : 3^N \longrightarrow \{win, \ lose\}$ such that:

- $(N, \emptyset, \emptyset)$ wins,
- $(\emptyset, \emptyset, N)$ loses,
- $S \sqsubseteq T$ and $V(S) = \text{wins}$ implies $V(T) = \text{wins}$.

A $(j,2)$ simple game (N, V) consists of a finite set N of voters together with a value function $V : J^N \longrightarrow \{win, \ lose\}$ such that:

- $(N, \emptyset, \dots, \emptyset)$ wins,
- $(\emptyset, \dots, \emptyset, N)$ loses,
- $S \sqsubseteq T$ and $V(S) = \text{wins}$ implies $V(T) = \text{wins}$.

Example

Consider the $(3,2)$ game defined on $N = \{1,2,3\}$ by its set of minimal winning tripartitions $W^m = \{(12,\emptyset,3),(1,23,\emptyset),(23,1,\emptyset)\}$ (keys and commas omitted for tripartitions).

$(123,\emptyset,\emptyset)$	winning but non-minimal
$(12,3,\emptyset)$	winning but non-minimal
$(13,2,\emptyset)$	winning but non-minimal
$(1,23,\emptyset)$	minimal winning
$(12,\emptyset,3)$	minimal winning
$(23,1,\emptyset)$	minimal winning
others	losing

Example (Absolute majority with abstention allowed)

Consider the absolute majority voting on $N = \{1, 2, 3\}$ defined by its set of minimal winning tripartitions $W^m = \{(12, \emptyset, 3), (13, \emptyset, 2), (23, \emptyset, 1)\}$ (keys and commas omitted for tripartitions).

$(123, \emptyset, \emptyset)$	winning but non-minimal
$(12, 3, \emptyset)$	winning but non-minimal
$(13, 2, \emptyset)$	winning but non-minimal
$(23, 1, \emptyset)$	winning but non-minimal
$(12, \emptyset, 3)$	minimal winning
$(13, \emptyset, 2)$	minimal winning
$(23, \emptyset, 1)$	minimal winning
others	losing

Example (Relative majority with abstention allowed)

Consider the relative majority voting on $N = \{1, 2, 3\}$ defined by its set of minimal winning tripartitions

$W^m = \{(12, \emptyset, 3), (13, \emptyset, 2), (23, \emptyset, 1), (1, 23, \emptyset), (2, 13, \emptyset), (3, 12, \emptyset)\}$

$(123, \emptyset, \emptyset)$	winning but non-minimal
$(12, 3, \emptyset)$	winning but non-minimal
$(13, 2, \emptyset)$	winning but non-minimal
$(23, 1, \emptyset)$	winning but non-minimal
$(12, \emptyset, 3)$	minimal winning
$(13, \emptyset, 2)$	minimal winning
$(23, \emptyset, 1)$	minimal winning
$(1, 23, \emptyset)$	minimal winning
$(2, 13, \emptyset)$	minimal winning
$(3, 12, \emptyset)$	minimal winning
others	losing

A $(3,2)$ simple game (N, W) is a weighted $(3,2)$ game if there exists a weight function $w : N \to \mathbb{R}^3$ and a quota q such that

$$S \in W \iff w(S) \geq q$$

where $w(S) = \sum_{p \in S_1} w^+(p) + \sum_{p \in S_3} w^-(p)$ for all $S = (S_1, S_2, S_3) \in 3^N$.

The only requirement for weights is:

$$w^+(p) \geq 0 \geq w^-(p)$$

Example

A resolution is carried in the Security Council if at least nine members support it and no permanent member is explicitly opposed. Let $P = \{1, 2, 3, 4, 5\}$ and $R = \{6, 7, ..., 15\}$ be resp. the set of permanent and nonpermanent members, and

$$V(S) = V(S_1, S_2, S_3) = \begin{cases} \text{win} & \text{if } |S_1| \geq 9 \text{ and } S_3 \cap P = \emptyset \\ \text{lose} & \text{otherwise} \end{cases}$$

This voting system with abstention can be represented as

$$[\ 9;\ \underbrace{(1, 0, -6), ..., (1, 0, -6)}_{5}, \underbrace{(1, 0, 0), ..., (1, 0, 0)}_{10}\ ]$$

Example

- The absolute majority voting rule with abstention is weighted and can be represented as

$$[\ \ 2;\ (1,0,0),(1,0,0),(1,0,0)\ \ ]$$

- The relative majority voting rule with abstention is weighted and can be represented as

$$[\ \ 1;\ (1,0,-1),(1,0,-1),(1,0,-1)\ \ ]$$

A partial order $\subseteq^j$ on the set J^N is considered. If $S, T \in J^N$, then $S \subseteq^j T$ means $S_k \subseteq^j \bigcup_{i=1}^{k} T_i$ for any $k = 1, \ldots, j-1$.

In words, S is contained in T if players in T are working in the same or in a higher level than in S. We use $S \subset^j T$ if $S \subseteq^j T$ and $S \neq T$.

The j-partition $\mathcal{N} = (\emptyset, \ldots, \emptyset, N)$ is the minimum for the order $\subseteq^j$.

Definition

Let N be a finite set and J^N be the set of all totally ordered j-partitions on N. A *j-cooperative game* is a function $v : J^N \to \mathbb{R}$ that assigns to every j-partition $S \subseteq N$ an attainable payoff $v(S)$ and fulfills: $v(\emptyset) = 0$ and $v(S) \leq v(T)$ if $S \subset^j T$ (monotonicity).

$$v(S) = \begin{cases} 4 - |S_2| - 2|S_3| & \text{if } a \in S_1 \\ \max\{0, |S_1| - |S_3|\} & \text{if } a \in S_2 \\ 0 & \text{if } a \in S_3 \end{cases}$$

where $S = (S_1, S_2, S_3)$ and S_1 contains the workers with full involvement, S_2 contains the workers with an intermediate involvement, and S_3 contains the rest of the workers with the lowest level of involvement.

If we do not have any information about workers' attitude, how should the total gain be distributed among them?

The value we propose assigns to them: $(2, 1, 1)$ where the payment 2 is for the qualified worker a.

$$\begin{aligned} v(abc, \emptyset, \emptyset) &= 4, \\ v(ab, c, \emptyset) = v(ac, b, \emptyset) &= 3, \\ v(a, bc, \emptyset) = v(ab, \emptyset, c) = v(ac, \emptyset, b) = v(bc, a, \emptyset) &= 2, \\ v(a, b, c) = v(a, c, b) = v(b, ac, \emptyset) = v(c, ab, \emptyset) &= 1, \\ \text{others} & \quad 0. \end{aligned}$$

$n = 5$ **players, the number of queues is** $5! \cdot 3^5 =$ **29160**:

Consider the queue given by the permutation 12345 and elections *nyaay*:

n	y	a	a	y
1	2	3	4	5

Players' marginal contributions:

Player	direct gain capacity		blocking capacity
$1 \rightarrow$			$v(12345, \emptyset, \emptyset) - v(2345, \emptyset, 1)$
$2 \rightarrow$	$v(2, \emptyset, 1345) - v(\emptyset, \emptyset, 12345)$		
$3 \rightarrow$	$v(2, 3, 145) - v(2, \emptyset, 1345)$	$+$	$v(2345, \emptyset, 1) - v(245, 3, 1)$
$4 \rightarrow$	$v(2, 34, 15) - v(2, 3, 145)$	$+$	$v(245, 3, 1) - v(25, 34, 1)$
$5 \rightarrow$	$v(25, 34, 1) - v(2, 34, 15)$		

- If $a \in S_j$:

$$S_{a\uparrow_k} = (S_1, \ldots, S_k \cup \{a\}, \ldots, S_j \setminus \{a\})$$

  for any $k = 1, \cdots, j-1$.

- If $a \in S_1$:

$$S_{a\downarrow_k} = (S_1 \setminus \{a\}, \ldots, S_k \cup \{a\}, \ldots, S_j)$$

  for any $k = 2, \cdots, j$.

The idea we pursue with these two definitions is to consider two special types of *marginal contributions* for j-partitions in a given game v:

$$\begin{array}{llll} m^k(v,S,a) = v(S_{a\uparrow_k}) - v(S) & \text{if} & a \in S_j \\ m_k(v,S,a) = v(S) - v(S_{a\downarrow_k}) & \text{if} & a \in S_1 \end{array}$$

Definition (A value for j-cooperative and $(j, 2)$-simple games)

For any $v \in \mathcal{J}_N$ and any player $a \in N$, the $\mathcal{F}$-*value* is defined as

$$\mathcal{F}_a(v) = \frac{1}{j^n n!}\left[\sum_{\substack{S \in J^N:\\ a \in S_j}} \sum_{k=1}^{j-1} \gamma_j^n(s_j - 1) m^k(v, S, a) + \sum_{\substack{S \in J^N:\\ a \in S_1}} \sum_{k=2}^{j} \gamma_j^n(s_1 - 1) m_k(v, S, a)\right] \quad (1)$$

where

$$\gamma_j^n(t) = t! j^t \sum_{i=0}^{t} \frac{(n - t - 1 + i)!}{j^i i!}, \quad (2)$$

for $t = 0, 1, \ldots, n - 1$.

$n \downarrow \mid t \rightarrow$	0	1	2	3	4	5
1	1					
2	1	3				
3	2	4	14			
4	6	10	22	90		
5	24	36	64	156	744	
6	120	168	264	504	1368	7560

Table: Numerical coefficients $\gamma_2^n(t)$ for 2-cooperative games up to 6 players.

$n \downarrow \mid t \rightarrow$	0	1	2	3	4	5
1	1					
2	1	4				
3	2	5	26			
4	6	12	36	240		
5	24	42	96	348	2904	
6	120	192	372	984	4296	43680

Table: Numerical coefficients $\gamma_3^n(t)$ for 3-cooperative games up to 6 players.

$n \downarrow \mid t \rightarrow$	0	1	2	3	4	5
1	1					
2	1	5				
3	2	6	42			
4	6	14	54	510		
5	24	48	136	672	8184	
6	120	216	504	1752	10872	163800

Table: Numerical coefficients $\gamma_4^n(t)$ for 4-cooperative games up to 6 players.

The tests for the new power index for j-simple games and for the value for j-cooperative games

1. - + Some axiomatizations exists but it is very difficult that all axioms be really appealing.
2. + I would say that the probabilistic model is quite compelling.
3. + - This index and this value potentially has a wide ranging applicability. But not yet sufficiently developed.
4. + -The index's computation is rather straightforward because we have a formulation. But, we have less alternative formulations and the complexity of computing the index or value increases with respect to simple games or cooperative games.

Open problems:

- Finding new axiomatizations. Appealing axioms (may be difficult)
- Application of the index / value to real problems.
- Look for some new formulations useful for computing the index / value when the number of voters / players is moderately high.
- Do extensions of the variants of the Shapley value to multichoice and $(j, 2)$-simple games. E.g., the Consensus Shapley value.
- Do a comparative analysis with other values that have been defined under the name of "Shapley value" and contrast their suitability.

References-I

1. Bernardi, G. and Freixas, J. (2018) The Shapley value analyzed under the Felsenthal and Machover bargaining model, Public Choice 176(3-4), 557–565.
2. Dubey, P. (1975) On the uniqueness of the Shapley value, Int. Journal of Game Theory 4(3), 131–139.
3. Einy, E. and Haimanko, O. (2011), Characterization of the Shapley-Shubik power index without the efficiency axiom, Games and Economic Behavior 73(2), 615–621.
4. Felsenthal, D.S. and Machover, M. (1996) Alternative forms of the Shapley value and the Shapley-Shubik index, Public Choice 87, 315–318.
5. Freixas, J. (2019) A value for j-cooperative games: some theoretical aspects and applications. Handbook of the Shapley value, edited by E. Algaba, V. Fragnelli and J. Sánchez-Soriano, Chapter 14, 281–311.

References-II

6. Freixas, J. and Zwicker, W.S. (2003) Weighted voting, abstention, and multiple levels of approval, Social Choice and Welfare 21(3), 399–431.
7. Hsiao, C.R. and Raghavan, T.E.S. (1992) Monotonicity and dummy free property for multi-choice cooperative games, Int. Journal of Game Theory 21, 301–302.
8. Hsiao, C.R. and Raghavan, T.E.S. (1993) Shapley value for multichoice cooperative games I, Games and Economic Behavior 5, 240–256.
9. Shapley, L.S. (1953) A value for *n*-Person Games. In: Kuhn, H.W., Tucker, A.W. (eds.): Contributions to the Theory of Games Vol.2, pp. 307–317, Princeton UP, Princeton.
10. Shapley, L.S. and Shubik, M. (1954) A method for evaluating the distribution of power in a committee system, The American Political Science Review 48(3), 787–792.

Ich danke für Ihre Aufmerksamkeit