Ordinal measures of influence in social structures

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Outline

Social ranking problem

Consider the director of a department who must provide a ranking over its professors reflecting their overall contributions over different research teams or groups.

The effect of cooperation among professors can be observed looking at the joint papers published by research teams (maybe combining different bibliometric indices...), the workshops and other scientific events organized together, the size of financed projects, and so on.

Thus, the starting point for an analysis is a ranking over groups

How can the director convert this information over groups into a ranking of individuals?

Basic notions and notations

A *binary relation R on a finite set X* is a subset of the Cartesian product $X \times X$. For each $x, y \in X$, the notation xRy will be preferably used instead of the more formal $(x, y) \in R$. A binary relation R is said to be:

- *reflexive*, if for each $x \in X$, xRx;
- *transitive*, if for each $x, y, z \in X$, xRy and $yRz \Rightarrow xRz$;
- *total*, if for each $x, y \in X$, $x \neq y \Rightarrow xRy$ or yRx;
- *symmetric*, if for each $x, y \in X$, $xRy \Leftrightarrow yRx$;
- *asymmetric*, if for each $x, y \in X$, $(x, y) \in R \Rightarrow (y, x) \notin R$;
- *antisymmetric*, if for each $x, y \in X$, xRy and $yRx \Rightarrow x = y$.

A reflexive, transitive and total binary relation on X is called a *total preorder* (also called, a *ranking*) on X.
An antisymmetric total preorder on X is called a *total order* on X.
$\mathcal{R}(X)$ denotes the set of rankings (or total preorders) on X.

Problem Definition

A finite set of individuals, alternatives, items...: $X = \{1, \ldots, |X|\}$

A total preorder $\succcurlyeq$ over $\mathcal{P}(X)$ (the set of all non-empty subsets of X):
$S \succcurlyeq T$: coalition $S \subseteq X$ is at least as strong as group $T \subseteq X$
($\sim$ the symmetric part, $\succ$ the asymmetric part).

A *social ranking solution* $R : \mathcal{R}(\mathcal{P}(X)) \rightarrow \mathcal{R}(X)$
that associates to every total preorder $\succcurlyeq$ over $\mathcal{P}(X)$ a total preorder $R(\succcurlyeq)$ or $R^{\succcurlyeq}$ over X.
($I^{\succcurlyeq}$ the symmetric part of $R^{\succcurlyeq}$, and $P^{\succcurlyeq}$ its asymmetric part).

Social ranking problem

A department with three professors $1, 2$ and 3. According to the opinion of the director, the effect of cooperation of the different research teams $S \subseteq X = \{1,2,3\}$ is as follows ($\succcurlyeq \in \mathcal{R}(\mathcal{P}(X))$):
$\{1,2,3\} \sim \{3\} \succ \{1,3\} \succ \{2\} \succ \{2,3\} \succ \{1\} \sim \{1,2\}$

Based on this information, the director asks us to make a ranking $R^{\succcurlyeq}$ over the three professors showing their attitude to collaborate with others as a team or autonomously.

Intuitively, 3 seems to be more influential than 1 and 2, as professor 3 belongs to the most successful teams in the above ranking.

Can we state more precisely the reasons driving us to the conclusion $3\ R^{\succcurlyeq}\ 1$ and $3\ R^{\succcurlyeq}\ 2$? and who between 1 and 2 is more "efficient"? $2\ R^{\succcurlyeq}\ 1$ or $1\ R^{\succcurlyeq}\ 2$?

Related literature

A set of "reasonable" properties that a social ranking solution $R : \mathcal{R}(\mathcal{P}(X)) \rightarrow \mathcal{R}(X)$ should satisfy: the lex-cel solution (Bernardi, Lucchetti, Moretti. *Soc Choice Welfare*,2019)

Other solutions: CP-majority (Haret, Khani, Moretti, Öztürk, *IJCAI2018*) and ordinal Banzhaf (Khani, Moretti, Öztürk, *IJCAI2019*); cardinality-based lex-cel (Algaba, Moretti, Rémila, Solal, 2020, submitted)

Manipulability of social rankings (Allouche, Escoffier, Moretti, Öztürk, *IJCAI2019*)

Other partial answer using invariant power indices (Moretti *Homo Oecon*,2015)

Axiom 1, Neutrality (N)

The social ranking of items in X should not depend on their names.

Definition

Let σ be a bijection on X. For any ranking $\succcurlyeq$ on $\mathcal{P}(X)$, let $\succcurlyeq_\sigma$ be the ranking given by

$$\sigma(S) \succcurlyeq_\sigma \sigma(T) \Leftrightarrow S \succcurlyeq T.$$

A solution R satisfies the *neutrality* property if

$$x \; R^{\succcurlyeq} \; y \quad \Leftrightarrow \quad \sigma(x) \; R^{\succcurlyeq_\sigma} \; \sigma(y)$$

for all $\succcurlyeq \in \mathcal{R}(\mathcal{P}(X))$ and $x, y \in X$.

For instance, a solution R that ranks the elements in $X = \{1, \ldots, |X|\}$ according to the ordering of their labels i.e., such that $xR(\succcurlyeq)y \Leftrightarrow x \geq y$ for every ranking in $\succcurlyeq \in \mathcal{R}(\mathcal{P}(X))$, does not satisfy the N property.

Axiom 2, Coalitional Anonymity (CA)

The relative ranking of two items i and j should only depend on their relative positions within groups containing either i or j but not both.

Definition

Suppose that for two rankings $\succcurlyeq, \sqsupseteq \in \boldsymbol{\mathcal{R}}(\mathcal{P}(X))$, there are two elements $x, y \in X$, and a bijection π on $2^{X\setminus\{x,y\}}$ such that, for all $S, T \in 2^{X\setminus\{x,y\}}$:

$$S \cup \{x\} \succcurlyeq T \cup \{y\} \Leftrightarrow \pi(S) \cup \{x\} \sqsupseteq T \cup \{y\}. \quad (1)$$

Then a solution R is *coalitional anonymous* if the following holds:

$$xR^{\succcurlyeq}y \Leftrightarrow xR^{\sqsupseteq}y.$$

CA: two linked principles

P.1) the position in the ranking of coalitions containing both 1 and 2, or neither 1 nor 2, does not influence the relative ranking between 1 and 2.

Example

Two rankings $\sqsupseteq, \geq \in \mathcal{R}(\mathcal{P}(X))$, with $X = \{1,2,3\}$, s.t.

$$\{1,2,3\} \sqsupset \{1\} \sqsupset \{2,3\} \sqsupset \{1,2\} \sqsupset \{2\} \sqsupset \{1,3\} \sqsupset \{3\}$$

$$\{1\} > \{1,2\} > \{2,3\} > \{1,2,3\} > \{3\} > \{2\} > \{1,3\}.$$

Let π be the bijection on $\{\emptyset, \{3\}\}$ s.t. $\pi(\emptyset) = \emptyset, \pi(\{3\}) = \{3\}$.

Condition (1) holds for $\sqsupseteq, \geq$ with the elements 1 and 2: $\geq$ differs from $\sqsupseteq$ *only* because of different positions of coalitions not containing them or containing both.
If R satisfies CA, then $1R^{\sqsupseteq}2 \Leftrightarrow 1R^{\geq}2$

CA: two linked principles

P.2) for the others coalitions, i.e. those which contain only one element between 1 and 2, it does not matter in which set 1 and 2 are, but uniquely their relative position in the ranking counts

Example

Consider the rankings $\succeq, \sqsupseteq \in \mathcal{R}(\mathcal{P}(X))$, with $X = \{1, 2, 3\}$ s.t.

$$\{1,2,3\} \succ \{1,3\} \succ \{2,3\} \succ \{1,2\} \succ \{2\} \succ \{1\} \succ \{3\}$$

$$\{1,2,3\} \sqsupset \{1\} \sqsupset \{2,3\} \sqsupset \{1,2\} \sqsupset \{2\} \sqsupset \{1,3\} \sqsupset \{3\}$$

Let π be the bijection on $\{\emptyset, \{3\}\}$ s.t. $\pi(\emptyset) = \{3\}, \pi(\{3\}) = \emptyset$.
Condition (1) is satisfied with 1 and 2 in the role of x and y, respectively. If R satisfies CA, then $1R^{\sqsupseteq}2 \Leftrightarrow 1R^{\succeq}2$

Some further notations

Suppose we have a ranking $\succcurlyeq \in \mathcal{R}(\mathcal{P}(X))$ of the form

$$S_1 \succcurlyeq S_2 \succcurlyeq S_3 \succcurlyeq \cdots \succcurlyeq S_{2^{|X|}-1}.$$

Given this ranking $\succcurlyeq$, we also consider its *quotient order*, denoted as follows

$$\Sigma_1 \succ \Sigma_2 \succ \Sigma_3 \succ \cdots \succ \Sigma_l$$

in which the subsets S_j are grouped in the *equivalence classes* Σ_k generated by the symmetric part of $\succcurlyeq$.

This means that all the sets in Σ_1 are indifferent to S_1 and are strictly better than the sets in Σ_2 and so on.

Two total preorders $\succcurlyeq, \sqsupseteq \in \mathcal{R}(\mathcal{P}(X))$ with the associated quotient orders:

$\Sigma_1 \succ \Sigma_2 \succ \cdots \succ \Sigma_u \succ \Sigma_{u+1} \succ \cdots \succ \Sigma_l$

$\Sigma_1 \sqsupset \Sigma_2 \sqsupset \cdots \sqsupset \quad \Sigma_u \cup \Sigma \quad \sqsupset \quad \Sigma_{u+1} \setminus \Sigma \quad \sqsupset \cdots \sqsupset \Sigma_l.$

$\{x, y\} \cap S = \{x\}$ for every $S \in \Sigma$

In $\sqsupseteq$ some subsets containing x but not y are strictly better ranked than in $\succcurlyeq$, and no subset containing y has changed its ranking position with respect to $\succcurlyeq$.

Definition

We say that $\sqsupseteq$ is *x-improving* and *y-invariant with respect to* $\succcurlyeq$.

Axiom 3: Monotonicity (M)

Definition

We say that a solution R is *monotone* if for any ranking $\succcurlyeq \in \mathcal{R}(\mathcal{P}(X))$, every $x, y \in X$ such that $xI^{\succcurlyeq}y$ and any ranking $\sqsupseteq \in \mathcal{R}(\mathcal{P}(X))$ which is x-improving and y-invariant with respect to $\succcurlyeq$, then it holds that $xP^{\sqsupseteq}y$.

Example

Consider a total preorder $\succcurlyeq \in \mathcal{R}(\mathcal{P}(X))$ such that

$$\{1,2,3\} \succ \{3\} \succ \{2\} \sim \{1,3\} \succ \{2,3\} \sim \{1\} \succ \{1,2\}$$

Suppose that $1I^{\succcurlyeq}2$.
Consider a total preorder $\sqsupseteq \in \mathcal{R}(\mathcal{P}(X))$ such that

$$\{1,2,3\} \sqsupset \{3\} \simeq \{1,3\} \sqsupset \{2\} \sqsupset \{2,3\} \simeq \{1\} \sqsupset \{1,2\}$$

Now, $1P^{\sqsupseteq}2$.

Axiom 4: Independence from the Worst Set (IWS)

Good groups are more important than the bad ones.

Definition

We say that a solution R is *independent of the worst set* if for any ranking $\succcurlyeq \in \mathcal{R}(\mathcal{P}(X))$ with the associated quotient order $\succ$ s.t. $\Sigma_1 \succ \Sigma_2 \succ \Sigma_3 \succ \cdots \succ \Sigma_l$ with $l \geq 2$, and $x, y \in X$ such that

$$xP^{\succcurlyeq}y,$$

then it holds

$$xP^{\sqsupseteq}y$$

for any partition $T_1, \ldots, T_m$ of Σ_l and for any ranking $\sqsupseteq \in \mathcal{R}(\mathcal{P}(X))$ with the associated quotient order $\sqsupset$ such that $\Sigma_1 \sqsupset \Sigma_2 \sqsupset \cdots \sqsupset \Sigma_{l-1} \sqsupset T_1 \sqsupset \cdots \sqsupset T_m$.

Example

Consider a total preorder $\succcurlyeq \in \boldsymbol{\mathcal{R}}(\mathcal{P}(X))$ such that

$$\{1,2,3\} \succ \{3\} \succ \{2\} \succ \{1,3\} \succ \{1\} \sim \{2,3\} \sim \{1,2\}$$

Suppose that $1P^{\succcurlyeq}2$.
Consider a total preorder $\sqsupseteq \in \boldsymbol{\mathcal{R}}(\mathcal{P}(X))$ such that

$$\{1,2,3\} \sqsupset \{3\} \sqsupset \{2\} \sqsupset \{1,3\} \sqsupset \{1\} \sqsupset \{2,3\} \sqsupset \{1,2\}$$

If R satisfies IWS, we still have $1P^{\sqsupseteq}2$.

Some notations

For any element $x \in X$, denote by x_k the number of sets containing x in the indifference class Σ_k, that is

$$x_k = |\{S \in \Sigma_k : x \in S\}|$$

for $k = 1, \ldots, l$. Let $\theta_{\succeq}(x)$ be the l-dimensional vector $\theta_{\succeq}(x) = (x_1, \ldots, x_l)$ associated to $\succeq$.

Now consider the lexicographic order among vectors:

$\mathbf{x} \geq_L \mathbf{y}$ if either $\mathbf{x} = \mathbf{y}$ or $\exists j : x_i = y_i, i = 1, \ldots, j-1 \wedge x_j > y_j$.

Lex-cel ranking solution

Definition

The *lexicographic excellence (lex-cel) solution* is the function $R_{\boldsymbol{le}} : \mathcal{R}(\mathcal{P}(X)) \to \mathcal{R}(X)$ defined for any ranking $\succcurlyeq \in \mathcal{R}(\mathcal{P}(X))$ as

$$xR_{\boldsymbol{le}}(\succcurlyeq)y \quad \text{if} \quad \theta_{\succcurlyeq}(x) \geq_L \theta_{\succcurlyeq}(y).$$

Example

Consider the initial total preorder
$\{1,2,3\} \sim \{3\} \succ \{1,3\} \succ \{2\} \succ \{2,3\} \succ \{1\} \sim \{1,2\}$

Σ_k	$\{1,2,3\}$, $\{3\}$	$\{1,3\}$	$\{2\}$	$\{2,3\}$	$\{1,2\}$, $\{1\}$	
$\theta_{\succeq}(1)$	1	1	0	0	2	
$\theta_{\succeq}(2)$	1	0	1	1	1	
$\theta_{\succeq}(3)$	2	1	0	1	0	

So the lex-cel ranking gives $3 P^{\succcurlyeq}_{\boldsymbol{le}}\ 1\ P^{\succcurlyeq}_{\boldsymbol{le}}\ 2$.

Characterization

Theorem [Bernardi, Lucchetti, Moretti (2018)]

The *lexicographic-excellence* (lex-cel) solution $R_{\boldsymbol{le}} : \boldsymbol{\mathcal{R}}(\mathcal{P}(X)) \rightarrow \boldsymbol{\mathcal{R}}(X)$ is the unique solution fulfilling axioms N, CA, M and IWS.

Remark

Axioms N, CA, M, and IWS are independent: they all are necessary in order to uniquely characterize the lexicographic excellence solution

If $\succcurlyeq$ is a total order, actually $R_{\boldsymbol{le}}(\succcurlyeq)$ provides an order and the $(2^n - 1)$-dimensional vector $\theta_{\succcurlyeq}(x)$ is boolean, i.e. made by only zeros and ones.

Example

Consider the total order

$$\{1,2,3\} \succ \{2\} \succ \{1,3\} \succ \{1,2\} \succ \{3\} \succ \{1\} \succ \{2,3\}$$

Σ_k	$\{1,2,3\}$	$\{2\}$	$\{1,3\}$	$\{1,2\}$	$\{3\}$	$\{1\}$	$\{2,3\}$
$\theta_{\succeq}(1)$	1	0	1	1	0	1	0
$\theta_{\succeq}(2)$	1	1	0	1	0	0	1
$\theta_{\succeq}(3)$	1	0	1	0	1	0	1

So the excellence ranking gives $2 \; P_{\boldsymbol{le}}^{\succcurlyeq} \; 1 \; P_{\boldsymbol{le}}^{\succcurlyeq} \; 3$.

Some remarks

For the lex-cel solution, excellence is rewarded. It is possible to define a solution dual to the lex-cel solution where mediocrity is punished.
Change one single axiom: from IWS to IBS (and from left-to-right to right-to-left, lexicographically...).

The axiomatic characterization requires the solution to be defined on the whole set of all total preorders on the subsets of X. What happens if we only consider total orders on $\mathcal{P}(X)$?

What if only total orders on $\mathcal{P}(X)$?

Definition (VIP)

We say that a solution R satisfies the *VIP* property if for any ranking $\succcurlyeq \in \mathcal{R}(\mathcal{P}(X))$ with associated quotient order

$$\Sigma_1 \succ \Sigma_2 \succ \Sigma_3 \succ \cdots \succ \Sigma_l$$

such that there exists $S^* \in \mathcal{P}(X)$ with $\Sigma_1 = \{S^*\}$, then $xP(\succcurlyeq)y$ for all $x \in S^*$ and all $y \in X \setminus S^*$.

Proposition

A solution $R : \mathcal{R}(\mathcal{P}(X)) \to \mathcal{R}(X)$ that satisfies axioms N, M and IWS also satisfies the VIP property.

Theorem

The lex-cel solution $R_{\boldsymbol{le}}$ is the unique solution fulfilling axioms CA and VIP on the class of total orders on $\mathcal{P}(X)$.

Ceteris Paribus (CP-) Majority (HKMO-IJCAI2018)

We compare two individuals $i, j \in X$ based on their relative contribution to groups of other individuals: $S \cup \{i\}$ vs. $S \cup \{j\}$ for all $S \subseteq X \setminus \{i, j\}$.

Example $X = \{1, 2, 3, 4\}$: $234 \succ 34 \succ 134$, $13 \succ 123 \succ 12 \succ 23$, $24 \succ 4 \succ 14$ (all the other coalitions are in a less preferred indifference class)

$S \subseteq X \setminus \{1, 2\}$	CP-comparison
$\emptyset$	$1 \sim 2$
3	$1\ 3 \succ 2\ 3$
4	$1\ 4 \prec 2\ 4$
34	$1\ 34 \prec 2\ 34$

Then, $2 P^{\succcurlyeq}_{CP} 1$

See HKMO-IJCAI2018 for an axiomatic characterization of the CP-majority rule (using properties similar to N, CA and M in the interpretation but on a different domain...).

Condorcet-like cycles

Example $X = \{1,2,3,4\}$: $234 \succ 34 \succ 134$, $13 \succ 123 \succ 12 \succ 23$, $24 \succ 4 \succ 14$ (all the other coalitions are in a less preferred indifference class)

The CP-majority $R^{\succcurlyeq}$ that:
$3P^{\succcurlyeq}2$ (since $13 \succ 12$ amd $34 \succ 24$),
$2P^{\succcurlyeq}1$ (since 2 beats 1 in$234 \succ 134$, $24 \succ 14$, and 1 beats 2 only in $13 \succ 23$),
$1P^{\succcurlyeq}3$ (since $12 \succ 23$ but $34 \succ 14$).

To avoid cycles using CP-comparisons:
1) Restricting the domain of coalitional relations (See HKMO-IJCAI 2018).
2) Weighting the role of CP-comparisons (See KMO-IJCAI 2019).

Weights for the CP-comparisons (KMO-IJCAI 2019)

Example $X = \{1,2,3,4\}$: $234 \succ 34 \succ 134$, $13 \succ 123 \succ 12 \succ 23$, $24 \succ 4 \succ 14$ (all the other coalitions are in a less preferred indifference class)

1 vs. 2	w^S_{12}	2 vs. 3	w^S_{23}	1 vs. 3	w^S_{13}
$13 \succ 23$	1	$12 \prec 13$	1	$12 \succ 23$	0
$14 \prec 24$	1	$24 \prec 34$	1	$14 \prec 34$	2
$134 \prec 234$	1				
	$2P^{\succsim}_W 1$		$3P^{\succsim}_W 2$		$3P^{\succsim}_W 1$

Compute the weight w^S_{ij} of the CP-comparison on $S \subseteq X \setminus \{i,j\}$ as the number of coalitions $\{S, S \cup \{i,j\}\}$ between $S \cup \{i\}$ and $S \cup \{j\}$

Weighted CP-majority

Theorem

The weighted CP-majority is a well-defined social ranking solution and coincides with the ordinal Banzhaf ranking.

Example $X = \{1,2,3,4\}$: $234 \succ 34 \succ 134,\ 13 \succ 123 \succ 12 \succ 23,\ 24 \succ 4 \succ 14$ (all the other coalitions are in a less preferred indifference class)

$S \in U_1$	$m_1^S(\succeq)$	$S \in U_2$	$m_2^S(\succeq)$	$S \in U_3$	$m_3^S(\succeq)$
1	0	2	0	3	0
12	1	12	1	13	1
13	1	23	1	23	1
14	−1	24	1	34	1
123	1	123	−1	123	1
124	−1	124	−1	134	1
134	−1	234	1	234	1
1234	−1	1234	−1	1234	−1
	$s_1^{\succeq} = -1$		$s_2^{\succeq} = 1$		$s_3^{\succeq} = 5$

So, $3P^{\succeq}2P^{\succeq}1$.

See KMO-IJCAI2019 for an axiomatic characterization of this rule.

Ranking professors

Example

Consider the initial total preorder
$\{1,2,3\} \sim \{3\} \succ \{1,3\} \succ \{2\} \succ \{2,3\} \succ \{1\} \sim \{1,2\} \succ \emptyset$

The lex-cel ranking gives (see previous slides) $3 P^{\succsim}_{\boldsymbol{le}}\ 1\ P^{\succsim}_{\boldsymbol{le}}\ 2$.

1 vs. 2	w^S_{12}	2 vs. 3	w^S_{23}	1 vs. 3	w^S_{13}
$1 \prec 2$	1	$2 \prec 3$	0	$1 \prec 3$	1
$13 \succ 23$	0	$12 \prec 13$	1	$12 \prec 23$	0
$2 I^{\succsim}_{CP} 1$	$2 P^{\succsim}_{W} 1$	$3 P^{\succsim}_{CP} 2$	$3 P^{\succsim}_{W} 2$	$3 P^{\succsim}_{CP} 1$	$3 P^{\succsim}_{W} 1$

The CP-majority gives $3 P^{\succsim}_{CP}\ 1\ I^{\succsim}_{CP}\ 2$
The ordinal Banzhaf gives $3 P^{\succsim}_{W}\ 2\ P^{\succsim}_{W}\ 1$

(from Algaba, Moretti, Remila, Solal (2020))

Analyse the performance of four attacking players of the Paris Saint Germain (PSG) team during the eight matches of Champions League played during the season 2019/2020 (before the break on March 2020 for the covid-19 emergency).

It is well known that the PSG coach Thomas Tuchel has to face a selection dilemma when he must select among the four attacking stars Di María (D), Icardi (I), Mbappé (M) and Neymar (N).

We considered all different subsets of the four stars, and we assessed some relevant parameters like the total number of points scored p, the number of goals scored s and the one of goals conceded c by those groups when employed together in a match.

coalitions	points (p)	goals scored (s)	goals conceded (c)
$\{I, D, M\}$	6	6	0
$\{I, D\}$	6	4	0
$\{I, M, N\}$	3	5	0
$\{D, N\}$	3	2	0
$\{M\}$	1	2	2
$\{N, M\}$	0	1	2

A coalitional ranking has been computed according to a lexicographic comparison of vectors (p, s, c)

$$\{I, D, M\} \succ \{I, D\} \succ \{I, M, N\} \succ \{D, N\} \succ \{M\} \succ \{N, M\} \succ S,$$

for each other $S \subseteq \{D, I, M, N\}$ (which are all in the same worst equivalence class).

Lex-cel ranking

$$\{I, D, M\} \succ \{I, D\} \succ \{I, M, N\} \succ \{D, N\} \succ \{M\} \succ \{N, M\} \succ S,$$

for each other $S \subseteq \{D, I, M, N\}$ (which are all in the same worst equivalence class).

Σ_k	$\{I, D, M\}$	$\{I, D\}$	$\{I, M, N\}$	$\{D, N\}$	$\{M\}$	$\{N, M\}$	other S
$\theta_{\succeq}(D)$	1	1	0	1	0	0	5
$\theta_{\succeq}(I)$	1	1	1	0	0	0	5
$\theta_{\succeq}(M)$	1	0	1	0	1	1	4
$\theta_{\succeq}(N)$	0	0	1	1	0	1	5

So, according to the lex-cel solution
Icardi $P^{\succeq}_{le}$ Di María $P^{\succeq}_{le}$ Mbappé $P^{\succeq}_{le}$ Neymar.

CP-majority

$$\{I, D, M\} \succ \{I, D\} \succ \{I, M, N\} \succ \{D, N\} \succ \{M\} \succ \{N, M\} \succ S,$$

for each other $S \subseteq \{D, I, M, N\}$ (which are all in the same worst equivalence class).

I vs. D	I vs. M	I vs. N	D vs. M
$I \sim D$	$I \prec M$	$I \sim N$	$D \prec M$
$IM \sim DM$	$DI \succ MD$	$DI \succ ND$	$DI \succ IM$
$IN \prec DN$	$IN \prec MN$	$IM \sim NM$	$DN \succ MN$
$IMN \succ DMN$	$DIN \sim DMN$	$DIM \succ NDM$	$DIN \prec IMN$
$I \; I^{\succsim}_{CP} \; D$	$M \; P^{\succsim}_{CP} \; I$	$I \; P^{\succsim}_{CP} \; N$	$D \; I^{\succsim}_{CP} \; M$

Not transitive!

Ordinal Banzhaf

$$\{I, D, M\} \succ \{I, D\} \succ \{I, M, N\} \succ \{D, N\} \succ \{M\} \succ \{N, M\} \succ S,$$

for each other $S \subseteq \{D, I, M, N\}$ (which are all in the same worst equivalence class).

I vs. D	w^S_{ID}	I vs. M	w^S_{IM}	I vs. N	w^S_{IN}	D vs. M	w^S_{DM}
$I \sim D$		$I \prec M$	2	$I \sim N$		$D \prec M$	2
$IM \sim DM$		$DI \succ MD$	1	$DI \succ ND$	0	$DI \succ IM$	1
$IN \prec DN$	2	$IN \prec MN$	1	$IM \sim NM$		$DN \succ MN$	2
$IMN \succ DMN$	2	$DIN \sim DMN$		$DIM \succ NDM$	2	$DIN \prec IMN$	2
	$I\ I^{\succcurlyeq}_W\ D$		$M\ P^{\succcurlyeq}_W\ I$		$I\ P^{\succcurlyeq}_W\ N$		$M\ P^{\succcurlyeq}_W\ D$

So, according to the ordinal Banzhaf solution
Mbappé $P^{\succcurlyeq}_W$ Di María $I^{\succcurlyeq}_W$ Icardi $P^{\succcurlyeq}_W$ Neymar.

Work in progress and future research

- other solutions (and other axioms)
- partial information
- strategic behaviour (e.g., manipulation)
- coalition formation
- applications

THeory and Evidence to Measure Influence in Social structures (THEMIS): 4-year project funded by the French National Research Agency (ANR) - starting March 2021.

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Thank you!

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