An Introduction to the Theory of Single-Winner Elections

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The main points

- many non-equivalent procedures are used for a seemingly same purpose
- all systems are based on some apparently plausible notion of winning
- each one of them has at least one major flaw
- there is fundamental asymmetry between proving the validity of a general principle and disproving it
- many results in the theory of voting take the form of an incompatibility between social choice desiderata
- some system reforms are based on social choice theory
- even these are not exempt from the incompatibilities
- overall, the intuition is not to be much trusted in designing and analyzing voting rules

Purely binary rules

- Copeland: conduct all $k(k-1)$ pairwise comparisons (typically using the majority rule) and count the number of victories of each alternative. This is the Copeland score of the alternative. The order of scores determines the Copeland ranking. (CE = Condorcet extension, CL = never elects the Condorcet loser)
- Dodgson: count for each alternative the minimum number of pairwise preference switches between adjacent alternatives in individual rankings for this alternative to become the Condorcet winner. The order of these minima determines the Dodgson ranking. (CE, non-CL)
- Max-min rule: for each alternative, determine its minimum support in all $(k-1)$ pairwise comparisons. The order of these minima determines the max-min ranking. (CE, non-CL)

Purely binary rules, cont'd

- Kemeny's median: generate all $k!$ rankings and determine the support for each pair in the ranking. The sum of supports determines the Kemeny ranking. (CE, CL)
- Scwartz's GOTCHA: determine the smallest set of alternatives which has the property that no alternative outside the set defeats in paired contest any alternative within the set. (CE, CL)
- Young's rule: determine for each alternative the minimum number of voters who would have to be ignored for this alternative to become the Condorcet winner. The order of these minima determines the Young ranking. (CE, CL)
- Successive elimination (aka amendment procedure): using a given agenda, conduct all $k - 1$ pairwise contests eliminating the losing alternative in each contest. The winner of the last contest is declared the overall winner. (CE, CL)

Purely position rules

- Plurality rule: each voter is assigned one vote, the sum of these determines the plurality score of each alternative. The plurality ranking is determined by the scores. (non-CE, non-CL)
- Alternative vote: if an alternative has the plurality score higher than 50% of the voters, it is elected. Otherwise, the alternative with the lowest plurality score is ignored in all rankings. If this results in a profile where some alternative has the plurality score that exceeds 50% of the voters, this alternative is elected. Otherwise one proceeda by eliminating alternatives until the plurality score threshold is exceeded. (non-CE, CL)
- Coombs' method: otherwise identical with the alternative vote, but the instead of eliminating the alternative with the smallest plurality score, one ignores the alternative ranked last by the largest number of voters. (non-CE, CL)

Purely positional rules, cont'd

- Bucklin's rule: same winning criterion as in alternative vote and Coombs' rule. The Bucklin score of an alternative at level *i* is the number of voters ranking it *i*th or higher. The Bucklin winner is the alternative whose Bucklin score at level *j* exceeds 50% of the voters, with the smallest value of *j*. (non-CE, non-CL)
- Majority judgment: each voter assigns an ordinal grade to each alternative, whereupon the median grade for each alternative is determined. The alternative with the highest median grade is the winner. (non-CE, non-CL)
- Range voting: the voters assign cardinal points (from a predetermined range of values) to alternatives and the alternative with the largest point sum is the winner. (non-CE, non-CL)

Purely positional rules, cont'd

- Approval voting: each voter assigns each alternative either 0 or 1 votes. The vote sum of each alternative determines its approval score and ranking of scores coincides with the approval ranking.(non-CE, non-CL)
- Negative plurality elimination rule: sequential rule where at each round the alternative with the largest number of last ranks is eliminated until just one alternative, the winner, remains. (non-CE, CL)

Rules that are both positional and binary

- Plurality runoff: conduct the plurality election. If the plurality winner has the score that exceeds 50% of the voters, this alternative is elected. Otherwise the plurality runoff winner is elected from the set of two alternatives using pairwise majority comparison.(non-CE, CL)
- Borda count: assign a Borda score to each alternatives so that each first rank gives $k-1$ points, each second rank gives $k-2$ points etc. Summing the points given by voters to an alternative indicates is Borda score. The Borda ranking is determined by the Borda scores. (non-CE, CL)
- Nanson's rule: sequential Borda elimination where at each round the alternatives with the average of smaller Borda scores are eliminated. (CE, CL)

Rules that are both, cont'd

- Baldwin's rule: sequential Borda elimination where at each round the alternative with the smallest Borda score is eliminated. (CE, CL)
- Black's method: choose the Condorcet winner when one exists, otherwise the Borda winner is elected. (CE, CL)

“We (all) are the champions” – or at least we can be

Five alternatives, five winners

4 voters	3 voters	2 voters
A	E	D
B	D	C
C	B	B
D	C	E
E	A	A

Table: 5 candidates, 5 winners

Plurality voting: A; plurality runoff voting: E, Condorcet extensions: D, Borda count: B; approval voting (with additional assumptions): C.

To an extent this holds for Condorcet extensions as well

. . . assuming, of course, that there is no Condorcet winner in the profile under scrutiny.

10 voters	7 voters	1 voter	7 voters	4 voters
D	B	B	C	D
A	C	A	A	C
B	A	C	B	A
C	D	D	D	B

Table: Discrepancy among some Condorcet extensions

Copeland: A, B, C
Dodgson: D
Max-min: D

The ambiguous majority principle

Condorcet extensions are based on pairwise majority comparisons. What if there are more than two candidates?

4 voter	3 voters	2 voters
A	B	C
C	C	B
B	A	A

Table: Ambiguous majority principle

1. Majority means receiving more votes than any other candidate $\Rightarrow$ A
2. Majority winner has to have the support of the absolute majority of voters, e.g. plurality runoff winner $\Rightarrow$ B
3. Majority winner is the same as the Condorcet winner $\Rightarrow$ C

Highest average ranking → Borda Count

Example

2 voters	2 voters	2 voters	1 voter
D	A	B	D
C	D	A	C
B	C	D	B
A	B	C	A

This yields the ranking DABC.
Now remove D. This gives: CBA, i.e. reversal of collective preference over A, B and C.
Fishburn: it is possible that the Borda winner wins in only one of the proper subsets of the alternative set.
Obviously, fiddling with the alternative set opens promising vistas for outcome control.

Discrepancy among positional procedures

2	2	2	3
A	A	C	D
B	D	B	B
C	C	D	C
D	B	A	A

Here the plurality winner is A, vote-for-two winner is B, vote-for-three winner is C, and the Borda winner D.

Theorem

Saari 1992. Consider the alternative set $c_1, \ldots, c_K$ of at least three elements. Then such a profile exists that alternative c_j wins when the voting rule is vote-for-j and this holds for $j = 1, \ldots, K-1$. Moreover, c_K is the Borda winner.

Does unanimity guarantee the same outcome?

No.

1 voter	1 voter	1 voter
A	A	A
B	B	B
C	C	C

Table: A unanimous profile

If vote-for-two system is used or approval voting with everyone approving of their two highest ranked alternatives, the outcome is an A-B tie, not A.

Pairwise victories $\rightarrow$ Condorcet extensions

Example

Condorcet's paradox

4 voters	4 voters	4 voters
A	B	C
C	A	B
B	C	A

Surely, there is no winner here, or what? If so, then removing this kind of "component" from any larger profile or adding it to some profile should not change the winners, right?

Surprise?

Example

A profile with a strong Condorcet winner

7 voters	4 voters
A	B
B	C
C	A

Adding the Condorcet paradox profile to this one results in a new Condorcet winner. N.B. the Borda winner remains the same in the 11- and 23-voter profiles.

Borda's paradox

Example

4 voters	3 voters	2 voters
A	B	C
B	C	B
C	A	A

Borda's points:

- plurality voting results in a bad outcome
- a superior system exists (Borda Count)

Condorcet's paradox

Example

1 voter	1 voter	1 voters
A	B	C
C	A	B
B	C	A

Surely, there is no winner here since the pairwise majority relation is cyclic. The customary pairwise comparison method (with $k-1$ comparisons for the set of k alternatives) does not distinguish this from the setting where there is a Condorcet winner.

Arrow's theorem

Theorem

Arrow'(1963): *No social welfare function satisfies the following conditions:*

1. *unrestricted domain (U)*
2. *independence of irrelevant alternatives (IIA)*
3. *Pareto (P)*
4. *non-dictatorship (D)*

Remark: social welfare functions assigns to each n-tuple of connected and transitive individual preference relations a (collective) connected and transitive preference relation.

Comments

- some interpretations are outlandish (‘collectively rational decisions cannot be reached by democratic voting procedures’)
- IIA is not a normative but a technical desideratum
- under suitable domain restrictions there are procedures that satisfy other conditions
- there is a system used in many contemporary parliaments (the amendment procedure) that fails not only on IIA but also on P (an example follows)

Comments, cont'd

Example

1 voter	1 voter	1 voter
A	B	D
B	D	C
D	C	A
C	A	B

Assume the following agenda of pairwise majority comparisons: 1. B vs D (B), 2. the winner vs. A (A), 3. the winner vs. C (C). This results in C. However, C is Pareto-dominated by D.

Gibbard-Satterthwaite theorem

Definition

A social choice function is manipulable (by individuals) iff there is a situation and an individual so that the latter can bring about a preferable outcome by preference misrepresentation than by truthful revelation of his/her preference ranking, *ceteris paribus*.

Definition

A social choice function is non-trivial (non-degenerate) iff for each alternative x, there is a preference profile so that x is chosen.

Gibbard-Satterthwaite theorem, cont'd

Theorem

*(*Gibbard-Satterthwaite 1973-75). Every universal and non-trivial resolute social choice function is either manipulable or dictatorial.

Comments

- practically all commonly used voting procedures may end up with a tie of two or more candidates, i.e. they are not resolute. Hence the theorem does not apply to them.
- manipulable procedures may be very difficult to manipulable.
- manipulation may backfire, i.e. lead to a inferior outcome than sincere voting.
- the information required for successful manipulation varies among procedures.

Smith's theorem

Definition

Point system is an aggregation method that assigns a certain real number to a candidate ranked first by a voter, a real number to a candidate ranked second etc. The points assigned to a candidate are then summed up to obtain the score of the candidate. The collective ranking of candidates is then obtained as the ranking of the scores.

Smith's theorem, cont'd

Definition

A point runoff system is any aggregation method where the collective ranks are determined as follows:

1. a monotonic point system is applied to all M candidates. Then a second monotonic point system is applied to the $M - K_1$ highest ranked candidates, etc. After some number of stages, the remaining K_r are ranked using a monotonic point system. This is the collective ranking of the K_r highest ranked candidates.
2. thereafter a candidate in the remaining set of candidates is assigned to the top position after the K_r best ones and the process is continues until all rankes are filled.

Smith's theorem, cont'd

Theorem

(Smith 1973). No point runoff system involving two or more stages and non-trivial point systems is monotonic. More precisely, if such a system determines the first place first, then a change of votes in a candidate's favor can remove him from the first place. If it determines the last place first, such a change can put a candidate in last place who was not previously there.

Comments

- practically all runoff systems are point runoff ones, i.e. the coverage of the theorem is quite extensive
- it includes not only the most common presidential elections system, but all kinds of short-listing procedures (see the next slide)
- nonmonotonicity is fundamentally at odds with the basic rationale of voting, viz. that additional support, *ceteris paribus*, should do no harm for the candidates

Comments, cont'd

Example

35	34	31		40	29	31
A	B	C		A	B	C
B	C	A	$\Rightarrow$	B	C	A
C	A	B		C	A	B

In the left-hand panel A wins in plurality runoff after C is eliminated. Suppose now that 5 of the 34 voters with BCA ranking increase the winner A's support by lifting A above both C and B, *ceteris paribus*, so that the profile of the right-hand panel ensues. Here C wins after B is eliminated. Hence additional support to the winner A makes it a non-winner.

Moulin's theorem

Theorem

(Moulin 1988) If there are more than three candidates and at least 25 voters, no voting rule satisfies both the Condorcet criterion and the participation condition.

Comments

- Moulin's theorem pertains to variable electorates (while e.g. Smith's pertains to fixed ones)
- F. Brandt et al. have more recently extended this result so that the lower bound of the number of voters is 12.
- the participation condition excludes the following two paradoxes:
  1. the no-show paradox which occurs when a group a identically-minded voters, when joining the electorate *ceteris paribus*, changes the outcome from X to Y where Y is the candidate the group ranks last (lowest).
  2. the more-is-less paradox which occurs when a group of identically-minded voters all ranking X first when joining the electorate *ceteris paribus*, changes the outcome from X to some other candidate (which they rank below X)

The proposal

The case for electoral reform in the U.S. as laid out by Maskin and Sen is based on the following argument:

- The plurality rule (in primaries and in elections at large) *may* result in Borda paradoxes, i.e. in winning candidates that would possibly be defeated by *all* other contestants in pairwise majority comparisons.
- If the voters were allowed to express their full preference rankings, the pairwise majority comparisons would sometimes lead to an indisputable winner (Condorcet) winner.
- Should a Condorcet winner not be present, the plurality runoff system might be used as a secondary (tie-breaking) device.

Critical remarks

Some questionable aspects:

1. The Condorcet winner is not necessarily the intuitively most plausible choice (see the next slide).
2. Condorcet extensions are vulnerable to various paradoxes that do not afflict some other procedures.
3. The absence of a Condorcet winner would, according to the proposal, leave us with a system that is not a Condorcet extension, but would suffer from a similar paradox as the Condorcet extensions

Critical remarks, cont'd

1 voter	*1 voter*	*1 voter*	*1 voter*	*1 voter*
D	E	C	D	E
E	A	D	E	B
A	C	E	B	A
B	B	A	C	D
C	D	B	A	C

Table: Fishburn's example

Here D is the Condorcet and E the Borda winner. The latter seems a more plausible choice than the former.

Critical remarks, cont'd

1 voter	*1 voter*	*1 voter*
A	B	C
D	D	D
C	A	B
B	C	A

Table: No voter ranks the Condorcet winner first

Ergo: it is not always clear that the Condorcet winner is the most plausible choice. It will be recalled that all Condorcet extensions are – by Moulin's theorem cited above – vulnerable to the no-show paradox. The plurality runoff procedure – which according to the proposal is to applied should there be no Condorcet winner – does not do much better.

Critical remarks, cont'd

6 voters	*5 voters*	*4 voters*	*2 voters*
A	C	B	B
B	A	C	A
C	B	A	C

Table: Plurality runoff is nonmonotonic

The outcome here is A, but if A gets some additional support, *ceteris paribus*, so that the 2 BAC voter become ABC voters, A is no longer the winner (C is).

Critical remarks, cont'd

The plurality runoff is also vulnerable the no-show paradox, i.e. a violation of the bottom property. This is shown here.

26	47	2	25
A	B	B	C
B	C	C	A
C	A	A	B

Table: Plurality runoff violates the bottom property

Here A wins. Let now those 47 voters with BCA ranking abstain, then C wins.

By way of concluding

- after more than 200 years the binary and positional intuitions about winning are still subject of controversy
- social choice desiderata are typically phrased as general statements applicable in all profiles
- the failure on a given desideratum thus consists of showing that a profile exists where the choice dictated (or excluded) by the desideratum is not made (is made) by the procedure
- finding out the prevalence of a given type of profiles may yield essential information about the importance of incompatibility results in practice
- some domain restrictions have been studied in detail (next slides), but much remains to be done.

domain	unrestricted		Condorcet	
procedure	C-winning	C-losing	C-winning	C-losing
amendment	1	1	1	1
Copeland	1	1	1	1
Dodgson	1	0	1	1
Kemeny	1	1	1	1
minimax	1	0	1	1
Schwartz	1	1	1	1
Young	1	1	1	1
Borda	0	1	0	1
plurality	0	0	0	0
Baldwin	1	1	1	1
Black	1	1	1	1
Bucklin	0	0	0	0
Coombs	0	1	0	1
Hare	0	1	0	1
Nanson	1	1	1	1
pl. runoff	0	1	0	1
approval	0	0	0	0
majority j.	0	0	0	0
range voting	0	0	0	0

10 Condorcet extensions and monotonicity failure in Condorcet domain

procedure	vulnerability to P-BOT	vulnerability to P-TOP
amendment	yes	no
maximin	no	no
Dodgson	yes	no
Nanson	yes	no
Baldwin	yes	no
Copeland	yes	no
Black	yes	no
Kemeny	yes	no
Schwartz	yes	no
Young	no	no

Table: 10 Condorcet extensions in Condorcet domains

paradox	subset choice		consistency	
domain	unrestr.	DSF	unrestr.	DSF
amendment	0	1	0	1
Copeland	0	1	0	1
Dodgson	0	1	0	1
Kemeny	0	1	0	1
minimax	0	1	0	1
Schwartz	0	1	0	1
Young	0	1	0	1
Borda	0	0	1	1
plurality	0	0	1	1
Baldwin	0	1	0	1
Black	0	1	0	1
Bucklin	0	0	0	0
Coombs	0	0	0	0
Hare	0	0	0	0
Nanson	0	1	0	1
pl. runoff	0	0	0	0
approval	1	1	1	1
majority j.	1	1	0	0
range voting	1	1	1	1

Table: Invulnerability to subset choice and consistency paradoxes under unrestricted and DSF domain

Some self-flagellation

Felix Brandt, Marie Matthäus, Christian Saile, Nov. 12, 2020: Felsenthal and Nurmi are incorrect in arguing

1. that Bucklin's method is invulnerable to the no-show paradox when the initial profile contains a Condorcet winner which is elected.
2. that Instant Runoff (Alternative Vote in our terminology) is invulnerable to the no-show paradox when the initial profile contains a Condorcet winner which is elected.

Both points of criticism are indisputably right.

Bucklin's method is vulnerable to the no show paradox under DSF constraint

3	2	1	1
A	B	E	B
D	C	C	A
B	E	A	E
E	A	D	C
C	D	B	D

Table: Bucklin's method results in a Condorcet winner

Here A is the Bucklin and Condorcet winner.
Now, remove the right-most voter. Now, B becomes the Bucklin winner.
Ergo: Brandt et al. are right, Felsenthal and Nurmi wrong.

Alternative vote is vulnerable to the no-show paradox under DSF constraint

5	4	3	2	2
B	A	C	D	D
A	B	A	C	B
D	D	B	A	A
C	C	D	B	C

Table: Alternative vote method results in a Condorcet winner

Here A is the Condorcet winner and is elected under alternative vote. With the two right-most voters abstaining, B wins (and they prefer B to A).
Ergo: Brandt et al. are right, Felsenthal and Nurmi wrong.

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