Values of cooperative games: marginalism, egalitarianism and implementation

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The notion of cooperative game

Definition (Coooperative game with side payments (TU game))

(also known as game with transferable utility (TU game))
is a pair (N, v) , *with*

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For any coalition S:
$v(S)$ (the worth of S) – is the joint payoff that the coalition S can guarantee its members by joint action, <u>without</u> any participation of the players from $N \setminus S$.

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(And then S can divide it among its members at will –
transferable utility ; side payments).

Cooperative games– some desirable properties

Definition

A cooperative game (N, v) *is*

- *monotone if* $U \supset T \;\; \Rightarrow \;\; v(U) \geq v(T)$
  ("larger coalitions can more");

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  ("cooperation is profitable – it pays to merge coalitions");
- *convex if for every pair of coalitions* $T, U \subset N$

$$v(T \cup U) + v(T \cap U) \geq v(T) + v(U)$$

  or equivalently – (N, v) *has the nondecreasing contributions property:*

$$\forall T, U, i \;\; (T \subset U, i \in T \cap U) \Rightarrow v(T) - v(T \setminus i) \leq v(U) - v(U \setminus i)$$

  $v(T) - v(T \setminus i)$ *is player* i*'s marginal contribution to coalition* T*.*

Example

$N = \{1, 2, 3, 4, 5\}$;

Coalition(s)	Worth	Coalition(s)	Worth
1	1	23 , 24 , 235 , 245	7
2	2	34 and 345	6
3 , 4	3	123 and 124	9
5	0	134 and 1345	8
12 and 125	6	135 and 145	5
13 and 14	5	234 and 2345	10
15	1	1235 and 1245	9
25	2	1234	15
35 and 45	3	N	15

In this game, player 5 is a null player : for every coalition T , $v(T \cup 5) = v(T)$, and players 3 and 4 are interchangeable :
for every $T \subseteq (N \setminus \{3, 4\})$, $v(T \cup 3) = v(T \cup 4)$.

Cooperative games: allocations and values

Definition (Allocation in a cooperative game)

An allocation in the game (N, v) *is any vector* $(x_1, x_2, \ldots x_n)$ *such that* $\sum_{i=1}^{n} x_i = v(N)$.

$=$ *The players somehow divide between themselves the worth of the grand coalition* N.

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Definition (Value)

A value *is any* one-element solution, *i.e., any function* ψ *assigning to every cooperative game* (N, v) *some allocation in this game,*

$\psi(v) = (\psi_1(v), \psi_2(v), \ldots \psi_n(v))$;

$\psi_j(v)$ – *value of player* j *in the game* v.

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$\psi_j(v)$ – value of player j in the game v.
Thus: A value is an allocation method (rule);

according to this definition, every value is efficient – $\sum_{i=1}^{n} \psi_i(v) = v(N)$.

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- Symmetry (**S**) :
  For every game (N, v) and every permutation π of the set N of players,

$$\psi_i(\pi^* v) = \psi_{\pi(i)}(v) \text{ for every } i \in N\,,$$

where $\pi^*(v)$ is the game defined by: $\pi^* v(S) = v(\pi(S))$ for each $S \subset N$.

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- Linearity (**L**) : For every two games $v, w \in \mathcal{G}_n$ and every constant c ,
  $\psi(v + w) = \psi(v) + \psi(w)$ and $\psi(c \cdot v) = c \cdot \psi(v)$.

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  $\psi(v + w) = \psi(v) + \psi(w)$ and $\psi(c \cdot v) = c \cdot \psi(v)$.
- Weak monotonicity (**WM**) : For every <u>monotone</u> game (N, v) , $\forall_{i \in N}$
  $\psi_i(v) \geq 0$
- Coalitional monotonicity (**CM**): For every coalition $T \subseteq N$ and every two games $v, w \in \mathcal{G}_n$ such that $(v(T) > w(T)$ and $v(S) = w(S) \quad \forall_{S \neq T})$ it holds that $\psi_i(v) \geq \psi_i(w) \quad \forall_{i \in T}$

The Shapley value

Theorem (Shapley)

The only value ψ satisfying (efficiency), equal treatment property, null player property and additivity is the **Shapley value**.

Definition (Shapley value)

The Shapley value , denoted by ϕ , is a function assigning to any cooperative game (N, v) the allocation $\phi(v) = (\phi_1(v), \phi_2(v), \dots \phi_n(v))$ in this game defined by

$$\phi_j(v) = \sum_{T \ni j} \frac{(t-1)!(n-t)!}{n!}(v(T) - v(T \setminus j)).$$

($n = \#N$, $t = \#T$, $\phi_j(v)$ – Shapley value of player j in the game v).

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Remark

The Shapley value is marginalistic – the value of a player is determined solely by his marginal contributions to coalitions.

The egalitarian value

Definition (Egalitarian value)

The egalitarian value e results from equal division of $v(N)$ among all players:

$$\forall k \;\; e_k(v) = \frac{v(N)}{n} \, .$$

Remark

Both values ϕ and e are symmetric, linear and weakly and coalitionally monotonic, but the egalitarian value clearly

- *is not marginalistic,*
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Theorem (van den Brink 2007)

The only additive value ψ having equal treatment and nullifying player properties ($(\forall_{T \ni j}\ v(T) = 0) \Rightarrow \psi_j(v) = 0$) is the egalitarian value, e.

"Reconciling marginalism with egalitarianism": Egalitarian Shapley values

Definition ("Egalitarian Shapley values" (Joosten 1996; van den Brink et al. 2013))

$$\epsilon_j(v) = \epsilon \cdot \phi_j(v) + (1-\epsilon)\frac{v(N)}{n}$$

($\epsilon \in [0\ ,\ 1]$ arbitrary but fixed)

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A characterization:

Theorem (Casajus and Huettner 2013)

The only additive values fulfilling

- *local monotonicity: if $v(S \cup i) \geq v(S \cup j)$ for every coalition S such that $i, j \notin S$, then $\psi_i(v) \geq \psi_j(v)$*
- *"null player in productive environment": if j is a null player in v and $v(N) \geq 0$, then $\psi_j(v) \geq 0$*

are egalitarian Shapley values.

Reconciling ff. : The solidarity value

Definition (The solidarity value (Nowak and Radzik 1994))

$$\sigma_k(v) = \sum_{T \ni k} \frac{(t-1)!(n-t)!}{n!} \sum_{j \in T} \frac{v(T) - v(T \setminus j)}{t}.$$

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(Extended to a parametrized family by Casajus and Huettner 2014, and embedded in a full-dimensional family by Béal, Rémila and Solal 2017).

Random order "implementation"

Shapley value: the probabilistic interpretation

Assuming that the grand coalition N is forming in a random order and all permutations π of players – i.e., orders in which they join the coalition – are equiprobable,
the number $\phi_j(v)$ – the Shapley value of player j – is the expected value of that player's marginal contribution to the coalition $H_{\pi,j}$ of his "predecessors" in the ordering:

$$m_{j,\pi}(v) = v(H_{\pi,j}) - v(H_{\pi,j} \setminus j)\,.$$

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2. and at the end the common pool is divided equally among all players.

General case: Common pool values

Definition (General common pool value)

A general common pool value on $\mathcal{G}_n$ is any value $\psi^{(Q)}$ defined by the formula

$$\psi_j^{(Q)}(v) = \mathbf{E}_\pi \left[q_{j,H_{\pi,j}} + \frac{1}{n} \sum_{k=1}^{n} (m_{k,\pi}(v) - q_{k,H_{\pi,k}}) \right]$$

where: $Q = \left((q_{k,S})_{k=1}^{n}\right)_{S \in 2^N, S \ni k}$;

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Interpretation

In any ordering π, each player (k) retains for himself the quantity $q_{k,H_{\pi,k}}$, and the rest of his marginal contribution (maybe negative!), $m_{k,\pi}(v) - q_k(v)$, goes to the common pool.

At the end, the common pool is divided equally between all players.

General common pool values: some examples

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Example

1. *Equal surplus division value (Driessen and Funaki 1991) :* $q_{j,S} \equiv v(j)$

$$\theta_k(v) = v(k) + \frac{1}{n} \left(v(N) - \sum_{j=1}^{n} v(j) \right) = \frac{v(N)}{n} + \frac{n-1}{n} v(k) - \frac{1}{n} \sum_{j \neq k} v(j)\,.$$

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2. *The ENSC value (Moulin 1985 ; Sun et al. 2017) :* $q_{j,S} \equiv v(N) - v(N \setminus j)$

$$\begin{aligned} \zeta_k(v) &= (v(N) - v(N \setminus k)) + \frac{1}{n} \left(v(N) - \sum_{i=1}^{n} (v(N) - v(N \setminus i)) \right) \\ &= \frac{v(N)}{n} + \frac{1}{n} \sum_{i \neq k} v(N \setminus i) - \frac{n-1}{n} v(N \setminus k) . \end{aligned}$$

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3. *The "ideal values" (Wang et al. 2018) :* $q_{j,S} \equiv \frac{\sum_{T \ni j} \mu_t v(T)}{2^{n-1}}$

A simple special case: Expectations-based values

Definition (Expectation-based value)

A value ψ is **expectations-based** *if, for any game (N, v), it is generated by a vector $h(v)$ of players' expectations according to the formula*

$$\psi_k^{(h)}(N, v) = h_k(v) + \frac{1}{n}\left(v(N) - \sum_{j=1}^{n} h_j(v)\right) \quad \forall\, k = 1, \ldots, n.$$

Remark

For every function of expectations h, the expectations-based value $\psi^{(h)}$ is a general common pool value.

Expectations-based values: an equivalent algorithm

Theorem (An alternative "procedural implementation")

For every game (N, v), **every** *vector $h(v)$ of players' expectations in this game and every player $k \in N$, this player's resulting expectations-based value*

$$\psi_k^{(h)}(N, v) = h_k(v) + \frac{1}{n}\left(v(N) - \sum_{j=1}^{n} h_j(v)\right)$$

is the expected value of player k's "portion" $c_{k,\pi}(v)$ in the permutation π, with

$$c_{k,\pi}(v) = \begin{cases} v(k) & + \quad \sum_{j \in N \setminus k} \frac{m_{j,\pi}(v) - h_j(v)}{\pi(j) - 1} & \pi(k) = 1 \\ h_k(v) & + \quad \sum_{j \in N \setminus H_{\pi,k}} \frac{m_{j,\pi}(v) - h_j(v)}{\pi(j) - 1} & \pi(k) > 1 \end{cases}$$

under the assumption of the grand coalition forming in a random order (all permutations π of players equiprobable).

Expectations-based values: an equivalent algorithm

Theorem (An alternative "procedural implementation")

*For every game (N, v), **every** vector $h(v)$ of players' expectations in this game and every player $k \in N$, this player's resulting expectations-based value*

$$\psi_k^{(h)}(N, v) = h_k(v) + \frac{1}{n}\left(v(N) - \sum_{j=1}^{n} h_j(v)\right)$$

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under the assumption of the grand coalition forming in a random order (all permutations π of players equiprobable).

Remark

Sun et al. (2017) proved a particular case of this theorem for the ENSC value.

Expectations-based values ff.

Remark

Every *expectations-based value* $\psi^{(h)}$ *on* $\mathcal{G}_n$ *is a general common pool value with* $q_{k,S} \equiv h_k(v)$.

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However, the class of general common pool values is seemingly too rich:

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Every efficient value ψ *is expectations-based (and so a general CPV)*

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However, the class of general common pool values is seemingly too rich:

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Every efficient value ψ *is expectations-based (and so a general CPV) – it is generated by the vector* $h(v) = \psi(v)$.

A restriction: Symmetric feasible common pool values

Definition (Feasible common pool value)

A common pool value $\psi^{(Q)}$ *on* $\mathcal{G}_n$ *is* *feasible* *if there exists a family of coefficients* $R = \left(\left(r_{k,\pi}\right)_{k=1}^{n}\right)_{\pi\in\Pi_N}$ *such that for every* k , v *and* π

$$0 \leq r_{k,\pi} \leq 1 \;\; \text{and} \;\; q_{k,H_{\pi,k}} = r_{k,\pi} \cdot m_{k,\pi}.$$

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A feasible common pool value $\psi^{(Q)}$ *on* $\mathcal{G}_n$ *is* *symmetric* *iff*

$$\forall \pi_1, \pi_2 \in \Pi_N \, \forall j, k \in N \ \ (\pi_1(j) = \pi_2(k) \Rightarrow r_{j,\pi_1} = r_{k,\pi_2}).$$

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So, for symmetric feasible CP values:

$$R = (r_1, r_2, \ldots, r_n) \ \ \textit{and} \ \ q_{j,H_{\pi,j}} \equiv r_{\pi(j)}.$$

Interpretation:

$r_k \in [0\ ,\ 1]$ is the proportion of own marginal contribution in the ordering π that a player coming as k-th in that ordering may retain.

Symmetric feasible CP values ff.

A symmetric feasible CPV (henceforth, wyrcommon pool value) on $\mathcal{G}_n$ with the coefficients $R = (r_1, r_2, \ldots, r_n)$ is given by

$$\begin{aligned}
\psi_j^{(R)}(v) &= \mathbf{E}_\pi \left[r_{\pi(j)} m_{j,\pi}(v) + \frac{1}{n} \left(v(N) - \sum_{k=1}^{n} r_{\pi(k)} m_{k,\pi}(v) \right) \right] \\
&= \sum_{\pi \in \Pi_N} \frac{r_{\pi(j)} m_{j,\pi}(v) + \frac{1}{n} \left(v(N) - \sum_{k=1}^{n} r_{\pi(k)} m_{k,\pi}(v) \right)}{n!}
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Example

1. *The Shapley value* ϕ *:* $\quad r_k \equiv 1 \quad \forall k \, \forall \pi$ *;*
2. *Egalitarian Shapley value* ϵ *:* $\quad r_k \equiv \epsilon \quad \forall k \, \forall \pi$ *;*

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3. *The solidarity value* σ *: coefficients difficult to compute*

Common pool values – properties

Properties

Every (symmetric feasible) common pool value ψ is:

- linear,
- weakly monotonic $(\psi(v) \geq \mathbf{0}$ for every monotone game $v)$,
- locally monotonic:
  if player j is "not weaker" than player k in the game v , ie.
  $\forall_{T \subseteq (N-\{i,j\})} v(T \cup j) \geq v(T \cup k)$,
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Corollary

All (symmetric feasible) common pool values are procedural.

"Procedures": Sharing MCs with predecessors

Scenario

1. The players arrive in a random order π; all orders (permutations of the set N) are equally probable.
2. Every arriving player, k, brings his marginal contribution, $m_{k,\pi}(v)$, to the coalition of his predecessors.

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Definition (Procedure)

A procedure s on $\mathcal{G}_n$ is a family of nonnegative coefficients $((s_{k,j})_{j=1}^{k})_{k=1}^{n}$ such that $(\forall k)\ \sum_{j=1}^{k} s_{k,j} = 1$.

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The procedural value ψ^s determined by the procedure s on $\mathcal{G}_n$ is defined by the formula

$$\psi_i^s(v) = \mathbf{E}_\pi \sum_{j \in N_{\pi,i}} s_{\pi(j),\pi(i)}\, m_{j,\pi}(v) = \sum_{\pi \in \Pi} \sum_{j \in N_{\pi,i}} \frac{s_{\pi(j),\pi(i)}\, m_{j,\pi}(v)}{n!}. \qquad (1)$$

($N_{\pi,j}$ is the set of successors of j in the ordering π , including j).

Some procedural values

Example

$\forall k\,(s_{k,k} = 1$ *and* $\forall j < k\,,\ s_{k,j} = 0)$ *– every player retains his entire marginal contribution to each coalition for himself* $\Longrightarrow$ *The Shapley value* ϕ

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$(\forall \, k \geq 1 \;\; \forall j \leq k) \;\; s_{k,j} = \frac{1}{k}$ $\Longrightarrow$ *The solidarity value* σ

Equivalent representations of procedures

Theorem (Equivalent representations)

If $s = ((s_{k,j})_{j=1}^{k})_{k=1}^{n}$ *and* $t = ((t_{k,j})_{j=1}^{k})_{k=1}^{n}$ *are two procedures such that for all k* $s_{k,k} = t_{k,k}$ *, then* $\psi^s = \psi^t$*.*

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Corollary

1. $s = (s_1, s_2, \ldots, s_n)$ *represents any procedure* $((s_{k,j})_{j=1}^{k})_{k=1}^{n}$ *on* $\mathcal{G}_n$ *, with* $s_{j,j} = s_j$ *for* $j = 1, 2, \ldots, n$

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Theorem (converse)

If $s = (s_1, s_2, \ldots, s_n)$ *and* $t = (t_1, t_2, \ldots, t_n)$ *are two different procedures on* $\mathcal{G}_n$*, then* $\psi^s \neq \psi^t$*.*

Procedural values: properties

Fact (Linearity of values with respect to procedure)

If $t = (1, t_2, \ldots, t_n)$ *and* $u = (1, u_2, \ldots, u_n)$ *are two procedures, then for every* $\lambda \in [0, 1]$ $\quad s = \lambda t + (1 - \lambda)u$ *is also a procedure,*
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- *weakly, coalitionally and locally monotonic.*

Procedural values: characterizations

Theorem

A value on $\mathcal{G}_n$ has the following properties:

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How about the converse?

Common pool and procedural values – relations ff.

By definition:

- Every symmetric feasible common pool value on $\mathcal{G}_n$ is defined by a sequence of coefficients $(r_1, r_2, \ldots, r_n)$, $\forall t \;\; r_t \in [0\ ,\ 1]$.
- Every procedural value on $\mathcal{G}_n$ is defined by a sequence of coefficients $(s_1, s_2, \ldots, s_n)$, $s_1 = 1\ ,\ \ \forall t > 1 \;\; s_t \in [0\ ,\ 1]$.

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But on the other hand:

- Different procedures (sequences $(s_1, s_2, \ldots s_n)$) always generate different values.
- Different sequences $(r_1, r_2, \ldots r_n)$ not necessarily generate different common pool values.

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But on the other hand:

- Different procedures (sequences $(s_1, s_2, \ldots s_n)$) always generate <u>different</u> values.
- Different sequences $(r_1, r_2, \ldots r_n)$ <u>not necessarily</u> generate different common pool values.

Example

The sequences $R = (\frac{1}{4}\ ,\ 1\ ,\ \frac{1}{4})$ *and* $R' = (\frac{3}{4}\ ,\ 0\ ,\ \frac{3}{4})$ *define the same common pool value on* $\mathcal{G}_3$ *– the egalitarian Shapley value with* $\epsilon = \frac{1}{2}$.

Equivalent forms of feasible common pool values

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Proposition

Let $R = (r_1, r_2, \ldots, r_n)$ *and* $R' = (r'_1, r'_2, \ldots, r'_n)$ *be two sequences of coefficients. Then, the symmetric feasible CPVs* $\psi^{(R)}$ *and* $\psi'^{(R)}$ *are equal if and only if*

- *either* $R = R'$
- *or* $r_t - r'_t = \prod_{u=1}^{t-1}\left(1 - \frac{n}{u}\right) \cdot (r_1 - r'_1) = (-1)^{t-1}\binom{n-1}{t-1}$ *for every* $t = 2, 3, \ldots, n$

Common pool and procedural values– relations ff.

Theorem

Every symmetric feasible common pool value on $\mathcal{G}_n$ given by the sequence of coefficients $R = (r_1, r_2, \ldots r_n)$ is a procedural value with coefficients

$$s_1 = 1 \ , \ \ s_t = \left(1 - \frac{t-1}{n}\right) r_{t-1} + \frac{t-1}{n} r_t \quad \textit{for } t = 2, 3, \ldots, n.$$

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Somewhat tedious but routine – using the Ruiz, Valenciano and Zarzuelo (1996) coefficients for linear and symmetric values.

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Corollary

Not all procedural values are symmetric feasible common pool values.

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Example

The procedural value defined by the sequence with $s_1 = 1$, $s_2 = 0$, $s_3 > \frac{2}{n}$ is not a (synnetric feasible) CPV:
$s_2 = 0 \ \Rightarrow \ r_1 = r_2 = 0 \ \Rightarrow \ s_3 = \frac{2}{n} \cdot r_3 \ \Rightarrow \ r_3 > 1.$

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Corollary

All symmetric feasible common pool values are procedural

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Not all procedural values are symmetric feasible CP values.

Extending the class of procedural values (1)

Definition (Extended procedural values)

Defined on $\mathcal{G}_n$ by sequence of triples of nonnegative coefficients:

$$(q, r, s) = (q_k, r_k, s_k)_{k=1}^{n}$$

such that $q_1 = r_n = 0$ and, for each k , $q_k + r_k + s_k = 1$.
Any player at k-th position in the ordering ($k = 1, 2, \ldots, n$) has to divide his marginal contribution in the following proportions:
– q_k jointly for all predecessors,
– r_k jointly for all successors, and
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Then, expectations over all equiprobable permutations are taken.

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Then, expectations over all equiprobable permutations are taken.

Theorem

Every linear, symmetric and weakly monotonic value on $\mathcal{G}_n$ is an extended procedural value defined by some extended procedure as above.

Extending the class of procedural values (2)

Definition (Ideal values Wang et al. 2018)

Defined on $\mathcal{G}_n$ by sequence $\mu = (\mu_1, \ldots, \mu_n)$ of nonnegative coefficients:
– μ_s – the share (every) player wants to grab from any $v(S)$ with $\#S = s$,
– $H_j^{(\mu)}(v) = \frac{\sum_{T \ni j} \mu_t v(T)}{2^{n-1}}$ – the average "demand" of player j,

$$\psi_j^{(\mu)}(v) = H_j^{(\mu)}(v) + \frac{1}{n}\left(v(N) - \sum_{k=1}^{n} H_k^{(\mu)}(v)\right)$$

– expectation-based values with expectations resulting from the demand coefficients.

Theorem (Wang et al. 2018)

Every linear, symmetric and coalitionally monotonic value on $\mathcal{G}_n$ is an ideal value defined by some family of demand coefficients as above.

New representatives: Generalized solidarity values

General solidarity values – scenario (Béal, Rémila and Solal 2017)

1. Players arrive in random order π to form the grand coalition.
2. As long as no more than p players are present (p fixed, $1 \leq p < n$), each retains his entire marginal contribution $m_{k,\pi}(v)$.

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Definition (p-solidarity values $\sigma^{(p)}$ (Béal, Rémila and Solal 2017))

For $p = 0, 1, 2, \ldots n-1$ *and for any player* $j \in N$,
$$\sigma_j^{(p)}(v) = \frac{1}{n!} \sum_{\pi \in \Pi} \frac{c_{j,\pi}(v)}{n!}$$

$$\textit{where} \quad c_{j,\pi}(v) = \begin{cases} m_{j,\pi}(v) \ \ (= v(H_{\pi,j}) - v(H_{\pi,j} \setminus j) & \textit{if } \pi(j) \leq p\,, \\ \frac{v(N) - v(\pi^{-1}(\{1,2,\ldots p\})}{n-p} & \textit{if } \pi(j) > p\,. \end{cases}$$

Generalized solidarity values ff.

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$$\sigma^{(0)} = e \ , \quad \sigma^{(n-1)} = \phi \, .$$

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A generalized "solidarity" value on $\mathcal{G}_n$ is any convex combination of values $\sigma^{(0)}, \ldots, \sigma^{(n-1)}$:

$$\sigma^{(\lambda)}(v) = \sum_{p=0}^{n-1} \lambda_p \sigma^{(p)}(v)$$

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More precisely: $\sigma^{(p)} = \psi^s$,
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1. $\mathcal{S}_n = \{\psi^s \in \mathcal{P}_n \, : \, 1 = s_1 \geq s_2 \geq \ldots \geq s_n\}$.
2. $\sigma \in \mathcal{S}_n$; *moreover,* σ *is the barycenter of* $\mathcal{S}_n$.

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Theorem (Axiomatic characterization (Béal, Rémila and Solal 2017))

A value ψ *on* $\mathcal{G}_n$ *is:*

• *linear,* • *weakly monotonic,* • *locally monotonic,*

and has the property of "null player in a null environment":
if j *is a null player in* v , $v(N) = 0$ *and* $v(S) \geq 0$ *for every* S, *then* $\psi_j(v) \leq 0$
if and only if it is a generalized solidarity value.

"Bidding for the surplus"

Algorithm (The bidding game (Perez-Castrillo and Wettstein 2001))

1. *Each player, say i, bids $n-1$ numbers $b_{i,j}$ to all other players for possibility of being a proposer in stage 3.*

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6. *If anyone rejects the proposal, then the proposer (p) receives $v(p)$ and leaves the game,*
7. *and all other players return to stage 1 in the game $v|_{N\setminus p}$.*

Equilibria of the "bidding for the surplus" game

Theorem (Perez-Castrillo and Wettstein 2001)

If the game (N, v) fulfils the condition $v(S \cup j) \geq v(S) + v(j) \;\; \forall S \, \forall j \notin S$, then

1. *the following joint strategy:*
   – each player i bids to each player $j \neq i$ the amount $b_{i,j} = \phi_j(v) - \phi_j(v|_{N \setminus i})$;
   – when a proposer, player k offers to each player $j \neq k$ the amount $y_{k,j} = \phi_j(v|_{N \setminus k})$;
   – when a responder, player j accepts an offer $z_{k,j}$ from k if and only if $z_{k,j} = \phi_j(v|_{N \setminus k})$
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2. *in every subgame perfect equilibrium, payoffs of all players are equal to their Shapley values in* (N, v)*.*

That is, the bidding game implements the Shapley value in subgame perfect Nash equilibria.

Some extensions

Remark

The implementation relies on the special properties of the Shapley value:

- $\forall_{i \in N} \;\; \phi_i(v) = \dfrac{v(N) - v(N \setminus i)}{n} - \dfrac{1}{n} \sum_{j \neq i} \phi_i(v|_{N \setminus j}),$
- $\forall_{i,j \in N} \;\; \phi_i(v) - \phi_i(v|_{N \setminus j}) = \phi_j(v) - \phi_j(v|_{N \setminus i})$ *(balanced contributions)*

However, some generalizations to other values are known:

- A similar mechanism, with a nonzero probability $1 - \epsilon$ of breakdown of negotiations in case of rejection (only at the stage when all n players were negotiating) and then all players receiving 0 implements the egalitarian Shapley value ϵ; (van den Brink, Funaki and Ju 2011)
- More elaborate three-stage mechanism implement the generalized solidarity values (Béal, Rémila and Solal 2017)

An infinite bargaining game

Algorithm (Basic setup (Hart and Mas-Colell 1986))

1. *One of the players is chosen at random to become a proposer.*
2. *The proposer, p, offers payments y_j satisfying $\sum_{j \neq p} y_j \leq v(N)$ to all other players.*

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4. *If all responders accept, each* $j \neq p$ *receives* y_j*, the proposer retains* $v(N) - \sum_{j \neq p} y_j$ *, and the game ends.*
5. *If anyone rejects the proposal, then:*
   – with probability $\rho < 1$ *the game moves back to stage 1,*
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Equilibrium of the bargaining game

Theorem (Hart and Mas-Colell)

If the game (N, v) is monotone, then

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4. *all players' payoffs in this equilibrium are equal to their Shapley values in (N, v).*

That is, the bargaining game **implements** the Shapley value in (unique) subgame perfect Nash equilibrium.

Remark

However, the equilibrium payoffs are <u>expectations</u> of random variables.

Implementing other values by the Hart - Mas-Colell game

When the game is modified by allowing for excluding another player instead of the proposer, other values are obtained as SP equilibrium payoffs.

Denoting:

- ρ – the probability of excluding a player after a rejection
- α – the (conditional) probability that the excluded player is the proposer,
- and assuming all responders to be excluded with the same probability,

we obtain e.g.

– the egalitarian value when $\alpha = 0$,

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and, generally, any bargaining value (Calvo and Guttierez-Lopez, forthcoming).

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