On the Shapley value of liability games

Based on joint work with Péter Csóka and Ferenc Illés (Corvinus)

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Bankruptcy problems from the Talmud

In the Talmud (Kethubot 93a) the following three "bankruptcy situations" and their "solutions" by Rabbi Nathan are recorded without any "explanation".

E	d_1	d_2	d_3		r_1	r_2	r_3
100	100	200	300	$\longrightarrow$	$100/3$	$100/3$	$100/3$
200	100	200	300	$\longrightarrow$	50	75	75
300	100	200	300	$\longrightarrow$	50	100	150

Supposedly there is some underlying "fairness principle".

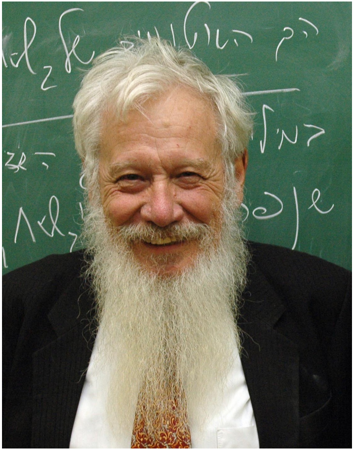
What is that principle? (equal split, proportional to claims, ???)

How the allocations were computed?

These questions puzzled Talmudic scholars for centuries.

Aumann and Maschler (1985) gave plausible explanations using cooperative game theory.

Aumann & Maschler (JET, 1985)



Robert J. Aumann (1930-)
Nobel Memorial Prize in Economics (2005)

Michael B. Maschler (1927-2008)
Frederick W. Lanchester Prize (1995)

Bankruptcy games

Barry O'Neill (MSS, 1982) investigated inheritance problems and Ibn Ezra's allocation rule. He associated with

a bankruptcy situation $(E; d_1, \ldots, d_n)$ with $E \leq d(N) = \sum_{i=1}^{n} d_i$

a bankruptcy game: $N = \{1, \ldots, n\}$ (player set)
$v(S) = \max\{0, E - d(N \setminus S)\}$ for all $S \subseteq N$ (coalitional function).

Using Talmudic principles O'Neill (1982) extended Ibn Ezra's rule to a random order rule that is induced by the Shapley value of the related "pessimistic" bankruptcy game.

Aumann and Maschler (1985) applied the nucleolus to the related bankruptcy games.

E	d_1	d_2	d_3		1	2	3	12	13	23	123
100	100	200	300	$\longrightarrow$	0	0	0	0	0	0	100
200	100	200	300	$\longrightarrow$	0	0	0	0	0	100	200
300	100	200	300	$\longrightarrow$	0	0	0	0	100	200	300

Bankruptcy rules via games

E	d_1	d_2	d_3
100	100	200	300
200	100	200	300
300	100	200	300

$\longrightarrow$

r_1	r_2	r_3
$100/3$	$100/3$	$100/3$
50	75	75
50	100	150

situation $\longrightarrow$ allocation

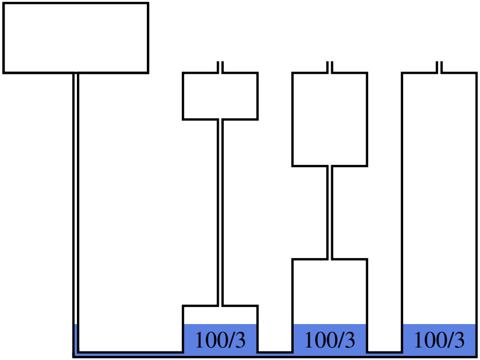
situation $\searrow$ game $\nearrow$ solution

game

1	2	3	12	13	23	123
0	0	0	0	0	0	100
0	0	0	0	0	100	200
0	0	0	0	100	200	300

The players who claim the whole estate are symmetric in the games and in the solutions. $\Longrightarrow$ The Talmud rule (i) ignores excessive claims; (ii) treats players with equal claims equally.

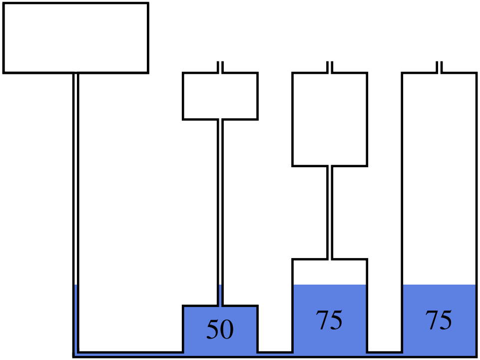
COMMUNICATING VESSELS (KAMINSKI, MSS, 2000)



Talmud-szabály (E = 100)

Graphics by Balázs Sziklai (Corvinus)

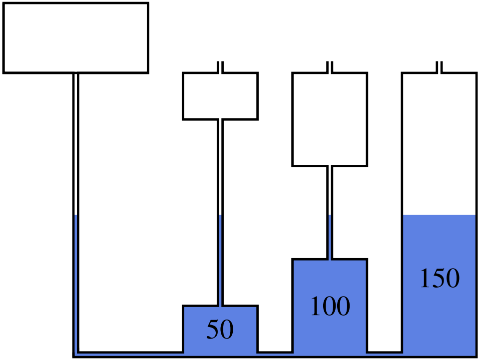
Communicating vessels (Kaminski, MSS, 2000)



Talmud-szabály (E = 200)

Graphics by Balázs Sziklai (Corvinus)

Communicating vessels (Kaminski, MSS, 2000)



Talmud-szabály (E = 300)

Graphics by Balázs Sziklai (Corvinus)

Liability problems

Insolvent firms (countries, households) often agree with their creditors to decrease the value of their liabilities. For sovereign defaults, about 30-40 % deficiency is documented by Benjamin and Wright (2009), D'Erasmo (2011), and Arslanalp and Henry (2005).

The question is how to distribute the asset value of the firm among the creditors and the firm itself.

Sturzenegger and Zettelmeyer (2007), Chatterjee and Eyigungor (2015) note that there is no settled theory for the renegotiations, but creditors may join ad hoc groups and try to force the firm to pay them first.

Our approach

$$
\begin{array}{cccc}
\text{rule:} & \text{situation} & \longrightarrow & \text{allocation} \\
 & \downarrow & & \uparrow \\
\text{value:} & \text{game} & \longrightarrow & \text{payoff vector}
\end{array}
$$

Liability games (Csóka and Herings, GEB, 2019)

The firm and the creditors are the players, any group of them can form their coalitions.

The worth of a coalition: what they can guarantee for themselves, irrespective of what outsiders do.

Given a coalition and the complementary coalition, the firm first pays as much as possible to the creditors in his coalition (up to their total liability value), then the firm pays the leftover to the complementary coalition.

$\longrightarrow$ Formal definition comes later.

Model assumptions: there is no outside authority that could “force” the firm to pay out everything but “the pressure to pay” cannot be ignored either. The firm might try to “exclude” some of the creditors, but they can form clubs to protect their interests. The model tries to capture the “bargaining powers” in the negotiations.

SOME BACKGROUND ON LIABILITY GAMES

Liability games are superadditive, but not additive, and constant-sum games: any coalition and its complement divide up the asset value.
$\Longrightarrow$ The core of a liability game is empty unless the firm is solvent.

Csóka and Herings (2019) investigated the nucleolus of liability games, and proved (among other things) that the nucleolus allocation rule satisfies

- efficiency: the asset value is fully divided up among the players
- non-negativity: no creditor pays, the firm has limited liability
- liabilities boundedness: no creditor receives more than his claim
- monotonicity in claims: higher claims bring higher payments, but also higher deficiencies

The insolvent firm keeps a positive amount but not more than half of the asset value.

Here we investigate the Shapley allocation rule.

TU GAMES

Transferable utility cooperative game (N, v)

- N is a non-empty, finite set of *players*
- $v : 2^N \to \mathbb{R}$ *coalitional function* with $v(\emptyset) = 0$

Game (N, v) is called

- *unanimity game* on $\emptyset \neq T \subseteq N$ if

$$v(S) = u_T(S) := \begin{Bmatrix} 1 & \text{if } S \supseteq T, \\ 0 & \text{otherwise.} \end{Bmatrix} \quad \text{for every } S \subseteq N$$

Denote by $\mathcal{G}^N$ the set of all games on fixed N. Let $n = |N|$.

REMARK

$\mathcal{G}^N$ is a linear vector space of dimension $2^n - 1$.

– game vector $v \in \mathbb{R}^{2^N}$ must satisfy $v(\emptyset) = 0$
– the $2^n - 1$ unanimity game vectors are linearly independent

VALUES FOR TU GAMES

A *value* on $\mathcal{G}^N$ is a map $f : \mathcal{G}^N \to \mathbb{R}^N$.

We say that value f satisfies

- *linearity*: if $f(\alpha v + \beta w) = \alpha f(v) + \beta f(w)$ holds for all $\alpha, \beta \in \mathbb{R}$ and $v, w \in \mathcal{G}^N$.
- *efficiency*: if $\sum_{j \in N} f_j(v) = v(N)$ holds for all $v \in \mathcal{G}^N$.
- *the equal treatment property*: if $j, k \in N$ are *symmetric* players in game $v \in \mathcal{G}^N$, that is if $v(S \cup j) = v(S \cup k)\ \forall S \subseteq N \setminus \{j, k\}$, then $f_j(v) = f_k(v)$.
- *the null player property*: if $j \in N$ is a *null* player in game $v \in \mathcal{G}^N$, that is if $v(S \cup j) - v(S) = 0\ \forall S \subseteq N \setminus j$, then $f_j(v) = 0$.

LLOYD SHAPLEY

Lloyd Shapley (1923-2016)

Nobel Memorial Prize in Economics (2012)

The Shapley value on $\mathcal{G}^N$

Theorem (Shapley, 1953)

The value $\phi : \mathcal{G}^N \to \mathbb{R}^N$ defined by

$$\phi_i(v) = \sum_{S \subseteq N \setminus i} \gamma_N(S)[v(S \cup i) - v(S)] \qquad (i \in N)$$

where $\gamma_N(S) = \dfrac{s!(n-1-s)!}{n!} = \dfrac{1}{n\binom{n-1}{s}}$ *and* $s = |S|, n = |N|$,

is the ONLY value on $\mathcal{G}^N$ that satisfies linearity, efficiency, the equal teatment property and the null player property.

Note: $\{\gamma_N(S)\}_{S \subseteq N \setminus i}$ is a probability distribution on $2^{N \setminus i}$ for any $i \in N$.

Interpretation: player i chooses partners randomly as follows:

1. choose the number s of partners uniformly from $0, 1, \ldots, n-1$;
2. choose the set of partners S with $|S| = s$ uniformly from $2^{N \setminus i}$.

Shapley value of the Talmud games

$E = 100$

1	2	3	12	13	23	123		ϕ_1	ϕ_2	ϕ_3
0	0	0	0	0	0	100	$\rightarrow$	$100/3$	$100/3$	$100/3$

$E = 200$

1	2	3	12	13	23	123		ϕ_1	ϕ_2	ϕ_3
0	0	0	0	0	100	200		$100/3$	$250/3$	$250/3$
0	0	0	0	0	100	100	$\rightarrow$	0	50	50
0	0	0	0	0	0	100		$100/3$	$100/3$	$100/3$

$E = 300$

1	2	3	12	13	23	123		ϕ_1	ϕ_2	ϕ_3
0	0	0	0	100	200	300		50	100	150
0	0	0	0	100	0	100	$\rightarrow$	50	0	50
0	0	0	0	0	200	200		0	100	100

THE SHAPLEY VALUE: UNIQUENESS

$$\phi\Big(v = \sum_{\emptyset \neq T \subseteq N} \alpha_T u_T\Big) = \sum_{\emptyset \neq T \subseteq N} \alpha_T \phi(u_T) \qquad \alpha_T = \sum_{\emptyset \neq R \subseteq T} (-1)^{|T \setminus R|} v(R)$$

EXAMPLE ($N = \{0, 1, 2\}$)

S	$\overline{0}$	$\overline{1}$	$\overline{2}$	$\overline{01}$	$\overline{02}$	$\overline{12}$	N		ϕ_0	ϕ_1	ϕ_2
v	v_0	v_1	v_2	v_{01}	v_{02}	v_{12}	v_N				
u_0	1	0	0	1	1	0	1	α_0	1	0	0
u_1	0	1	0	1	0	1	1	α_1	0	1	0
u_2	0	0	1	0	1	1	1	α_2	0	0	1
u_{01}	0	0	0	1	0	0	1	α_{01}	$1/2$	$1/2$	0
u_{02}	0	0	0	0	1	0	1	α_{02}	$1/2$	0	$1/2$
u_{12}	0	0	0	0	0	1	1	α_{12}	0	$1/2$	$1/2$
u_{012}	0	0	0	0	0	0	1	α_{012}	$1/3$	$1/3$	$1/3$

NEUMANN AND MORGENSTERN (TGEB, 1944)

John von Neumann (1903-1957)

Oskar Morgenstern (1902-1977)

Constant-sum games

Game (N, v) is called

- *constant-sum* if $v(S) + v(N \setminus S) = v(N)$ for every $S \subseteq N$;

Denote by $\mathcal{G}^N_{CS}$ the set of constant-sum games on fixed N.

Remark

$\mathcal{G}^N_{CS}$ is a linear vector space of dimension $\leq 2^{n-1}$.

- game vector $v \in \mathbb{R}^{2^N}$ must satisfy 2^{n-1} independent equations
  $v(\emptyset) + v(N \setminus \emptyset) = v(N)$ implies $v(\emptyset) = 0$.

Remark

Unanimity game u_T is constant-sum if and only if $|T| = 1$ (a dictator game $u_{\{i\}}$ for some $i \in N$).

$\boxed{?}$ Find a "suitable" basis for $\mathcal{G}^N_{CS}$.

A BASIS FOR CONSTANT-SUM GAMES

Arbitrarily choose a player, $0 \in N$. Let $C = N \setminus \{0\}$.
Denote by $\mathcal{P}_0 = \{S \subseteq N : 0 \in S\}$ the set of partner coalitions of 0, and by $\mathcal{C}_0 = \{S \subseteq N : 0 \notin S\}$ the complement coalitions.
Clearly, $|\mathcal{P}_0| = |\mathcal{C}_0| = 2^{n-1}$.

Define for $0 \in R \subsetneq N$ the constant-sum game $d^R \in \mathcal{G}^N_{CS}$ by

$$d^R(S) = \left\{ \begin{array}{ll} 1 & \text{if } S = R \\ -1 & \text{if } S = N \setminus R \\ 0 & \text{otherwise} \end{array} \right\} \quad \text{for all } S \subseteq N. \tag{1}$$

For $R = N$, the constant-sum game $d^N \in \mathcal{G}^N_{CS}$ is defined as

$$d^N(S) = \left\{ \begin{array}{ll} 1 & \text{if } S = N \text{ or } 0 \notin S \neq \emptyset \\ 0 & \text{otherwise} \end{array} \right\} \quad \text{for all } S \subseteq N. \tag{2}$$

3-PLAYER CONSTANT-SUM GAMES

EXAMPLE ($n = 3$, $N = \overline{0} \cup \overline{12}$)

	$\mathcal{P}_0$				$\mathcal{C}_0$			
S	$\overline{0}$	$\overline{01}$	$\overline{02}$	N	$\varnothing$	$\overline{1}$	$\overline{2}$	$\overline{12}$
v	$v_{\overline{0}}$	$v_{\overline{01}}$	$v_{\overline{02}}$	v_N	0	$v_N - v_{\overline{02}}$	$v_N - v_{\overline{01}}$	$v_N - v_{\overline{0}}$
$d^{\overline{0}}$	1	0	0	0	0	0	0	-1
$d^{\overline{01}}$	0	1	0	0	0	0	-1	0
$d^{\overline{02}}$	0	0	1	0	0	-1	0	0
d^N	0	0	0	1	0	1	1	1

Trivially, for each $v \in \mathcal{G}_{CS}^{\overline{012}}$, the unique decomposition:

$$v = v_{\overline{0}} \cdot d^{\overline{0}} + v_{\overline{01}} \cdot d^{\overline{01}} + v_{\overline{02}} \cdot d^{\overline{02}} + v_N \cdot d^N .$$

By linearity, $\phi(v) = v_{\overline{0}} \cdot \phi(d^{\overline{0}}) + v_{\overline{01}} \cdot \phi(d^{\overline{01}}) + v_{\overline{02}} \cdot \phi(d^{\overline{02}}) + v_N \cdot \phi(d^N)$.

A basis for constant-sum games /2

Proposition

The games $d^R \in \mathcal{G}^N_{CS}$ ($R \in \mathcal{P}_0$) form a basis of $\mathcal{G}^N_{CS}$, henceforth $\dim(\mathcal{G}^N_{CS}) = 2^{n-1}$.

Moreover, $v(S) = \sum_{R \in \mathcal{P}_0} v(R) \cdot d^R(S)$ *for all* $S \subseteq N$ *and* $v \in \mathcal{G}^N_{CS}$.

Consequently, by linearity of the Shapley value,

$\phi(v) = \sum_{R \in \mathcal{P}_0} v(R) \cdot \phi(d^R)$.

Note: no null player in any of the basic games d^R ($R \in \mathcal{P}_0$).
Efficiency and the equal treatment property only determine a value on d^R upto a parameter, say $f_0(d^R)$.

3-PLAYER CONSTANT-SUM GAMES /2

EXAMPLE ($n = 3$, $N = \overline{0} \cup \overline{12}$)

Let value f be linear, efficient, and satisfy equal treatment.

	$\mathcal{P}_0$							
S	$\overline{0}$	$\overline{01}$	$\overline{02}$	N				
v	$v_{\overline{0}}$	$v_{\overline{01}}$	$v_{\overline{02}}$	v_N	$\longrightarrow$	$f_0(v)$	$f_1(v)$	$f_2(v)$
$d^{\overline{0}}$	1	0	0	0	$v_{\overline{0}}$	$f_0^{\overline{0}}$	$-f_0^{\overline{0}}/2$	$-f_0^{\overline{0}}/2$
$d^{\overline{01}}$	0	1	0	0	$v_{\overline{01}}$	$f_0^{\overline{01}}$	$f_0^{\overline{01}}$	$-2f_0^{\overline{01}}$
$d^{\overline{02}}$	0	0	1	0	$v_{\overline{02}}$	$f_0^{\overline{02}}$	$-2f_0^{\overline{02}}$	$f_0^{\overline{02}}$
d^N	0	0	0	1	v_N	f_0^N	$(1 - f_0^N)/2$	$(1 - f_0^N)/2$

THE SHAPLEY VALUE OF CONSTANT-SUM GAMES

PROPOSITION (SHAPLEY, 1953)

Let $v \in \mathbb{R}^{2^N}$ be constant-sum: $v(S) = v(N) - v(N \setminus S)$ for all $S \in 2^N$. The Shapley payoff to $i \in N$ is

$$\phi_i(v) = -v(N) + 2 \sum_{S \subseteq N \setminus i} \gamma_N(S) v(S \cup i)$$

$$\phi_i(v) = -v(N) + \frac{2}{n} \sum_{S \subseteq N \setminus i} \frac{v(S \cup i)}{\binom{n-1}{s}}$$

where $s = |S|, n = |N|$.

Note: the Shapley payoff to player i depends on the average values of the same-size coalitions containing i, no need to compute the marginal contributions of i.

Shapley value for 3-player constant-sum games

Example ($n = 3$, $N = \overline{0} \cup \overline{12}$)

S	$\mathcal{P}_0$: $\overline{0}$	$\overline{01}$	$\overline{02}$	N				
v	$v_{\overline{0}}$	$v_{\overline{01}}$	$v_{\overline{02}}$	v_N	$\longrightarrow$	$\phi_0(v)$	$\phi_1(v)$	$\phi_2(v)$
$d^{\overline{0}}$	1	0	0	0	$v_{\overline{0}}$	$2/3$	$-1/3$	$-1/3$
$d^{\overline{01}}$	0	1	0	0	$v_{\overline{01}}$	$1/3$	$1/3$	$-2/3$
$d^{\overline{02}}$	0	0	1	0	$v_{\overline{02}}$	$1/3$	$-2/3$	$1/3$
d^N	0	0	0	1	v_N	$-1/3$	$2/3$	$2/3$

In formula,

$$\phi_0(v) = \frac{2v_{\overline{0}} + v_{\overline{01}} + v_{\overline{02}} - v_N}{3} \qquad \phi_i(v) = \frac{-v_{\overline{0}} + v_{\overline{0i}} - 2v_{\overline{0j}} + 2v_N}{3} \qquad (i \neq j)$$

LIABILITY PROBLEM

$N = \{0, 1, \ldots, c\}$ is the set of agents, where

- 0 is a *firm* having *asset value* $A \geq 0$ and
- a set of *creditors* $C = \{1, \ldots, c\}$, each with a *liability* $\ell_i \geq 0$

DEFINITION

A *liability problem* on $N = \{0\} \cup C$ is a pair $(A, \ell) \in \mathbb{R}_+ \times \mathbb{R}_+^C$ with $A \leq \ell(C) = \sum_{i \in C} \ell_i$.

Notation:

$\mathcal{L}^N$ class of liability problems on $N = \{0\} \cup C$

$\ell_S = \ell(S) = \sum_{i \in S} \ell_i$ total liabilities for subset of creditors $S \subseteq C$

$\ell_S^A = \min\{A; \ell_S\}$ truncated total liabilities for creditor group $S \subseteq C$

$\ell_i^A = \min\{A; \ell_i\}$ truncated liability to creditor $i \in C$

LIABILITY RULE

DEFINITION

A *liability rule* is a function that associates with each $(A, \ell) \in \mathcal{L}^N$ a unique payment vector $f = f(A, \ell) \in \mathbb{R}_+ \times \mathbb{R}_+^C$ satisfying

- Non-negativity. The firm has limited liability: $f_0 \geq 0$, and no creditor should be asked to pay: $f_i \geq 0$ for all $i \in C$,
- Liabilities boundedness. No creditor should be paid more than his claim: $f_i \leq \ell_i$ for all $i \in C$, and
- Efficiency. The sum of payments should be equal to the asset value: $f_0 + \sum_{i \in C} f_i = A$.

Notice that the above three properties imply:

$$0 \leq f_0 \leq A \quad \text{and} \quad 0 \leq f_i \leq \ell_i^A \leq \ell_i \text{ for all } i \in C.$$

Liability game

Definition

On $N = \{0\} \cup C$, liability problem $(A, \ell) \in \mathcal{L}^N$ induce *liability game* $v \in \mathbb{R}^{2^N}$ defined for all $S \in 2^N$ as follows:

$$v(S) = \begin{cases} \min\{A; \ell(S \setminus \{0\})\}, & \text{if } 0 \in S, \\ \max\{0; A - \ell(C \setminus S)\}, & \text{if } 0 \notin S. \end{cases}$$

Remark

$v(\emptyset) = 0 = v(\{0\}), \qquad 0 \leq v(S) \leq A \ \ \forall S \subseteq N, \qquad v(N) = A$

$i = 0$ *null player iff* $\ell_C = A \qquad i \in C$ *null player iff* $\ell_i = 0$

Liability problems (A, ℓ) *and* (A, ℓ^A) *induce the same liability game.*

Proposition (Csóka, Herings, 2019)

v constant-sum, superadditive, monotonic

- *v additive if and only if $\ell(C) = A$, the firm is solvent.*
- *If $\ell(C) > A$ (the firm is insolvent), the core is empty.*

THE SHAPLEY RULE (1, 2 CREDITORS)

EXAMPLE ($n = 2$)

S	$v(S)$	ϕ_0	ϕ_1
$\bar{0}$	$0\cdot$	$2 \cdot 1/2 = 1$	-1
N	$A\cdot$	$-1 + 2 \cdot 1/2 = 0$	1

$\phi_0 = 0; \quad \phi_1 = A$

EXAMPLE ($n = 3; \quad \ell_i^A = \min\{A, \ell_i\}$)

S	$v(S)$	ϕ_0	ϕ_1	ϕ_2
$\bar{0}$	$0\cdot$	$2 \cdot 1/3 = 2/3$	$-1/3$	$-1/3$
$\overline{01}$	$\ell_1^A\cdot$	$2 \cdot 1/6 = 1/3$	$1/3$	$-2/3$
$\overline{02}$	$\ell_2^A\cdot$	$2 \cdot 1/6 = 1/3$	$-2/3$	$1/3$
N	$A\cdot$	$-1 + 2 \cdot 1/3 = -1/3$	$2/3$	$2/3$

$$\phi_0 = \frac{\ell_1^A + \ell_2^A - A}{3}; \qquad \phi_i = \phi_0 + (A - \ell_j^A) \qquad i \neq j \in C$$

The Shapley rule (3 creditors)

Example ($n = 4$; $\ell_i^A = \min\{A, \ell_i\}$; $\ell_{ij}^A = \min\{A, \ell_i + \ell_j\}$)

S	$v(S)$		ϕ_0	ϕ_1	ϕ_2	ϕ_3
$\overline{0}$	$0\cdot$	$2 \cdot 1/4 =$	$3/6$	$-1/6$	$-1/6$	$-1/6$
$\overline{01}$	$\ell_1^A\cdot$	$2 \cdot 1/12 =$	$1/6$	$1/6$	$-1/6$	$-1/6$
$\overline{02}$	$\ell_2^A\cdot$	$2 \cdot 1/12 =$	$1/6$	$-1/6$	$1/6$	$-1/6$
$\overline{03}$	$\ell_3^A\cdot$	$2 \cdot 1/12 =$	$1/6$	$-1/6$	$-1/6$	$1/6$
$\overline{012}$	$\ell_{12}^A\cdot$	$2 \cdot 1/12 =$	$1/6$	$1/6$	$1/6$	$-3/6$
$\overline{013}$	$\ell_{13}^A\cdot$	$2 \cdot 1/12 =$	$1/6$	$1/6$	$-3/6$	$1/6$
$\overline{023}$	$\ell_{23}^A\cdot$	$2 \cdot 1/12 =$	$1/6$	$-3/6$	$1/6$	$1/6$
N	$A\cdot$	$-1 + 2 \cdot 1/4 =$	$-3/6$	$3/6$	$3/6$	$3/6$

$$\phi_0 = \frac{\sum_i \ell_i^A + \sum_{ij} \ell_{ij}^A - 3A}{6}; \qquad \phi_i = \phi_0 + \frac{3A - \ell_j^A - \ell_k^A - 2\ell_{jk}^A}{3}$$

The Shapley rule (4 creditors)

Example ($n = 5$; $\quad \ell^A_{ijk} = \min\{A, \ell_i + \ell_j + \ell_k\}$)

v_P	0	ℓ^A_1	ℓ^A_2	ℓ^A_3	ℓ^A_4	ℓ^A_{12}	ℓ^A_{13}	ℓ^A_{14}	ℓ^A_{23}	ℓ^A_{24}	ℓ^A_{34}	ℓ^A_{123}	ℓ^A_{124}	ℓ^A_{134}	ℓ^A_{234}	A
ϕ_0	$\frac{2}{5}$	$\frac{1}{10}$	$\frac{1}{10}$	$\frac{1}{10}$	$\frac{1}{10}$	$\frac{1}{15}$	$\frac{1}{15}$	$\frac{1}{15}$	$\frac{1}{15}$	$\frac{1}{15}$	$\frac{1}{15}$	$\frac{1}{10}$	$\frac{1}{10}$	$\frac{1}{10}$	$\frac{1}{10}$	$\frac{-3}{5}$
ϕ_1	$\frac{-1}{10}$	$\frac{1}{10}$	$\frac{-1}{15}$	$\frac{-1}{15}$	$\frac{-1}{15}$	$\frac{1}{15}$	$\frac{1}{15}$	$\frac{1}{15}$	$\frac{-1}{10}$	$\frac{-1}{10}$	$\frac{-1}{10}$	$\frac{1}{10}$	$\frac{1}{10}$	$\frac{1}{10}$	$\frac{-2}{5}$	$\frac{2}{5}$
ϕ_2	$\frac{-1}{10}$	$\frac{-1}{15}$	$\frac{1}{10}$	$\frac{-1}{15}$	$\frac{-1}{15}$	$\frac{1}{15}$	$\frac{-1}{10}$	$\frac{-1}{10}$	$\frac{1}{15}$	$\frac{1}{15}$	$\frac{-1}{10}$	$\frac{1}{10}$	$\frac{1}{10}$	$\frac{-2}{5}$	$\frac{1}{10}$	$\frac{2}{5}$
ϕ_3	$\frac{-1}{10}$	$\frac{-1}{15}$	$\frac{-1}{15}$	$\frac{1}{10}$	$\frac{-1}{15}$	$\frac{-1}{10}$	$\frac{1}{15}$	$\frac{-1}{10}$	$\frac{1}{15}$	$\frac{-1}{10}$	$\frac{1}{15}$	$\frac{1}{10}$	$\frac{-2}{5}$	$\frac{1}{10}$	$\frac{1}{10}$	$\frac{2}{5}$
ϕ_4	$\frac{-1}{10}$	$\frac{-1}{15}$	$\frac{-1}{15}$	$\frac{-1}{15}$	$\frac{1}{10}$	$\frac{-1}{10}$	$\frac{-1}{10}$	$\frac{1}{15}$	$\frac{-1}{10}$	$\frac{1}{15}$	$\frac{1}{15}$	$\frac{-2}{5}$	$\frac{1}{10}$	$\frac{1}{10}$	$\frac{1}{10}$	$\frac{2}{5}$

$$\phi_0 = \frac{3\sum_i \ell^A_i + 2\sum_{ij} \ell^A_{ij} + 3\sum_{ijk} \ell^A_{ijk} - 18A}{30} \qquad \leq \frac{3}{5}A$$

Properties of the Shapley rule /1

The Shapley rule is a liability rule. Thus,

- it satisfies efficiency, non-negativity and (truncated) liabilities boundedness.
- as any liability rule, respects minimal rights of creditors, i.e. it satisfies $\phi_i \geq \max\{0, A - \ell(C \setminus i)\}$ for any $i \in C$.
- as any rule induced by a solution of the associated liability game, ignores excessive parts of claims, i.e. $\phi(A, \ell) = \phi(A, \ell^A)$.

Bounds on the Shapley payments

For the firm

$$0 \leq \frac{n-2}{n} \min\{A, \min_{i \in C} \ell_i, \ell_C - A\} \leq \phi_0(A, \ell) \leq \frac{n-2}{n} A.$$

For creditor $i \in C$

$$0 \leq \frac{2}{n(n-1)} \ell_i^A \leq \phi_i(A, \ell) \leq \ell_i^A$$

All bounds are sharp.

PROPERTIES OF THE SHAPLEY RULE /2

$(A, \ell) \in \mathcal{L}^N$ a liability problem, v the induced liability game.

PROPOSITION (ORDER PRESERVATION)

If $\ell_i \geq \ell_j$ for creditors $i, j \in C$, then $\phi_i \geq \phi_j$ and $\ell_i - \phi_i \geq \ell_j - \phi_j$.

COROLLARY (EQUAL TREATMENT)

If $\ell_i = \ell_j$ for creditors $i, j \in C$, then $\phi_i = \phi_j$.

PROPOSITION (SUPERMODULARITY)

Let (A, ℓ) and (A', ℓ) be such that $\ell(C) \geq A' > A$. If $\ell_i \geq \ell_j$ for creditors $i, j \in C$ then

$$\phi_i(A', \ell) - \phi_i(A, \ell) \geq \phi_j(A', \ell) - \phi_j(A, \ell).$$

Properties of the Shapley rule /3

Proposition (Liabilities monotonicity)

Let (A, ℓ) and (A, ℓ') be such that $\ell'_i > \ell_i$ for $i \in C$, and $\ell'_k = \ell_k$ for all $k \in C \setminus i$. Then

$$\phi_i(A, \ell') \geq \phi_i(A, \ell) + \frac{2}{n(n-1)} \min\{\ell'_i - \ell_i, A - \ell_i^A\}.$$

Moreover, $\phi_0(A, \ell') \geq \phi_0(A, \ell)$.

Proposition (Asset monotonicity)

Let (A, ℓ) and (A', ℓ) be such that $\ell(C) \geq A' > A$. Then for any creditor $i \in C$,

$$\phi_i(A', \ell) \geq \phi_i(A, \ell).$$

But, $\phi_0(A', \ell)$ can be higher / lower than $\phi_0(A, \ell)$.

Computational complexity of the Shapley rule

Theorem

It is NP-hard to compute the Shapley-rule payment to the insolvent firm (from the parameters of the liability problem).

"Explanation": Already generating the liability game takes exponential time (in terms of the size of the liability problem). The Shapley value takes all marginal contributions into account with positive probabilities. Computing all marginal contributions takes expontial time (in terms of the size of the liability game).

Our approach

$$\begin{array}{ccc} \text{liability problem} & \longrightarrow & \text{Shapley allocation} \\ \downarrow & & \uparrow \\ \text{liability game} & \longrightarrow & \text{Shapley value} \end{array}$$

THANK YOU

Questions?

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